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ON THE PERFECT BLACKNESS OF THE CENTRAL SPOT IN NEWTON'S RINGS, AND ON THE VERIFICATION OF FRESNEL'S FORMULÆ FOR THE INTENSITIES OF REFLECTED AND REFRACTED RAYS.

By G. G. Stokes.

When Newton's rings are formed between two glasses of the same kind, the central spot in the reflected rings is observed to be perfectly black. This result is completely at variance with the theory of emissions, according to which the central spot ought to be half as bright as the brightest part of the bright rings, supposing the incident light homogeneous. On the theory of undulations, the intensity of the light reflected at the middle point depends entirely on the proportions in which light is reflected and refracted at the two surfaces of the plate of air, or other interposed medium, whatever it may be. The perfect blackness of the central spot was first explained by Poisson, in the case of a perpendicular incidence, who shewed that when the infinite series of reflections and refractions is taken into account, the expression for the intensity at the centre vanishes, the formula for the intensity of light reflected at a perpendicular incidence first given by Dr. Young being assumed. Fresnel extended this conclusion to all incidences by means of a law discovered experimentally by M. Arago, that light is reflected in the same proportions at the first and second surfaces of a transparent plate.* I have thought of a very simple mode of obtaining M. Arago's law from theory, and at the same time establishing theoretically the loss of half an undulation in internal, or else in external reflection.

* See Dr. Lloyd's Report on Physical Optics.—*Reports of the British Association*, vol. III. p. 344.

This method rests on what may be called the *principle of reversion*, a principle which may be enunciated as follows.

If any material system, in which the forces acting depend only on the positions of the particles, be in motion, if at any instant the velocities of the particles be reversed, the previous motion will be repeated in a reverse order. In other words, whatever were the positions of the particles at the time t before the instant of reversion, the same will they be at an equal interval of time t after reversion; from whence it follows that the velocities of the particles in the two cases will be equal in magnitude and opposite in direction.

Let S be the surface of separation of two media which are both transparent, homogeneous, and uncrystallized. For the present purpose S may be supposed a plane. Let A be a point in the surface S where a ray is incident along IA in the first medium. Let AR, AF be the directions of the reflected and refacted rays, AR' the direction of the reflected ray for a ray incident along FA, and therefore also the direction of the refracted ray for a ray incident along RA. Suppose the vibrations in the incident ray to be either parallel or perpendicular to the plane of incidence. Then the vibrations in the reflected and refracted rays will be in the first case parallel and in the second case perpendicular to the plane of incidence, since everything is symmetrical with respect to that plane. The direction of vibration being determined, it remains to determine the alteration of the coefficient of vibration. Let the maximum vibration in the incident light be taken for unity, and, according to the notation employed in Airy's *Tract*, let the coefficient of vibration be multiplied by b for reflection and by c for refraction at the surface S, and by e for reflection and f for refraction at a parallel surface separating the second medium from a third, of the same nature as the first.

Let x be measured from A negatively backwards along AI, and positively forwards along AR or AF, and let it denote the distance from A of the particle considered multiplied by the refractive index of the medium in which the particle is situated, so that it expresses an equivalent length of path in vacuum. Let λ be the length of a wave, and v the velocity of propagation in vacuum; and for shortness' sake let

$$\frac{2\pi}{\lambda}(vt - x) = X.$$

Then $\sin X$, $b \sin X$, $c \sin X$ may be taken to represent respectively the incident, reflected, and refracted rays; and

it follows from the principle of reversion, if we suppose it applicable to light, that the reflected and refracted rays reversed will produce the incident ray reversed. Now if in the reversed rays we measure x positively along AI or AR', and negatively along AR or AF, the reflected ray reversed will give rise to the rays represented by

$b^2 \sin X$, reflected along AI;

$bc \sin X$, refracted along AR';*

and the refracted ray reversed will give rise to

$cf \sin X$, refracted along AI;

$ce \sin X$, reflected along AR'.

The two rays along AR' superposed must destroy each other, and the two along AI must give a ray represented by $\sin X$. We have therefore

$$bc + ce = 0, \quad b^2 + cf = 1;$$

and therefore, since c is not zero,

$$b = -e \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$cf = 1 - b^2 = 1 - e^2 \quad \ldots\ldots\ldots\ldots (2).$$

Equation (1) contains at the same time M. Arago's law and the loss of half an undulation; and equations (1) and (2) together explain the perfect blackness of the centre of Newton's rings. (See Airy's *Tract.*)

If the incident light be common light, or polarized light, of any kind except plane polarized for which the plane of

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* It does not at once appear whether on reversing a ray we ought or ought not to change the sign of the coefficient; but the following considerations will shew that we must leave the sign unaltered. Let the portion of a wave, in which the displacement of the ether is in the direction which is considered positive, be called the *positive portion*, and the remaining part the *negative portion*; and let the points of separation be called *nodes*. There are evidently two sorts of nodes: the nodes of one sort, which may be called *positive nodes*, being situated in front of the positive portions of the waves, and the nodes of the other sort, which may be called *negative nodes*, being situated behind the positive portions or in front of the negative, the terms *in front* and *behind* referring to the direction of propagation. Now when the angle X vanishes, the particle considered is in a node; and since, at the same time, the expression for the velocity of the particle is positive, the coefficient of $\sin X$ being supposed positive, the node in question is a positive node. When a ray is reversed, we must in the first instance change the sign of the coefficient, since the velocity is reversed; but since the nodes which in the direct ray were positive are negative in the reversed ray, and *vice versâ*, we must moreover add $\pm \pi$ to the phase, which comes to the same thing as changing the sign back again. Thus we must take $b^2 \sin X$, as in the text, and not $-b^2 \sin X$, to represent the ray reflected along AI, and so in other cases.

polarization either coincides with the plane of incidence or is perpendicular to it, we can resolve the vibrations in and perpendicular to the plane of incidence, and consider the two parts separately.

It may be observed that the principle of reversion is just as applicable to the theory of emissions as to the theory of undulations; and thus the emissionists are called on to explain how two rays incident along RA, FA respectively can fail to produce a ray along AR'. In truth this is not so much a new difficulty as an old difficulty in a new shape; for if any mode could be conceived of explaining interference on the theory of emissions, it would probably explain the non-existence of the ray along AR'.

Although the principle of reversion applies to the theory of emissions, it does not lead, on that theory, to the law of intensity resulting from equations (1) and (2). For the formation of these equations involves the additional principle of superposition, which on the theory of undulations is merely a general dynamical principle applied to the fundamental hypotheses, but which does not apply to the theory of emissions, or at best must be assumed, on that theory, as the expression of a property which we are compelled to attribute to light, although it appears inexplicable.

In forming equations (1) and (2) it has been tacitly assumed that the reflections and refractions were unaccompanied by any change of phase, except the loss of half an undulation, which may be regarded indifferently as a change of phase of 180°, or a change of sign of the coefficient of vibration. In very highly refracting substances, however, such as diamond, it appears that when the incident light is polarized in a plane perpendicular to the plane of incidence, the reflected light does not wholly vanish at the polarizing angle; but as the angle of incidence passes through the polarizing angle, the intensity of the reflected light passes through a small minimum value, and the phase changes rapidly through an angle of nearly 180°. Suppose, for the sake of perfect generality, that all the reflections and refractions are accompanied by changes of phase. While the coefficient of vibration is multiplied by b, c, e, or f, according to the previous notation, let the phase of vibration be accelerated by the angle β, γ, ε, or ϕ, a retardation being reckoned as a negative acceleration. Then, if we still take $\sin X$ to represent the incident ray, we must take $b \sin(X+\beta)$, $c \sin(X+\gamma)$ to represent respectively the reflected and the refracted rays. After reversion we must change the signs

of β and γ, because, whatever distance a given phase of vibration has receded from A in consequence of the acceleration accompanying reflection or refraction, the same additional distance will it have to get over in returning to A after reversion. We have therefore $b \sin(X - \beta)$, $c \sin(X - \gamma)$ to represent the rays incident along RA, FA, which together produce the ray $\sin X$ along AI. Now the ray along RA alone would produce the rays

$$b^2 \sin X \text{ along } AI, \quad bc \sin(X - \beta + \gamma) \text{ along } AR';$$

and the ray along FA alone would produce the rays

$$cf \sin(X - \gamma + \phi) \text{ along } AI, \quad ce \sin(X - \gamma + \epsilon) \text{ along } AR'.$$

We have therefore in the same way as before,

$$cf \sin(X - \gamma + \phi) = (1 - b^2) \sin X,$$

$$b \sin(X - \beta + \gamma) + e \sin(X - \gamma + \epsilon) = 0.$$

Now each of these equations has to hold good for general values of X, and therefore, as may very easily be proved, the angles added to X in the two terms must either be equal or must differ by a multiple of 180°. But the addition of any multiple of 360° to the angle in question leaves everything the same as before, and the addition of 180° comes to the same thing as changing the sign of c or f in the first equation, or of b or e in the second. We are therefore at liberty to take

$$\phi = \gamma \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

$$\beta + \epsilon = 2\gamma \ldots\ldots\ldots\ldots\ldots(4);$$

and the relations between b, c, e, and f will be the same as before. Hence M. Arago's law holds good even when reflection and refraction are accompanied by a change of phase.

Equations (3) and (4) express the following laws with reference to the changes of phase. *The sum of the accelerations of phase at the two reflections is equal to the sum of the accelerations at the two refractions; and the accelerations at the two refractions are equal to each other.* It will be observed that the accelerations are here supposed to be so measured as to give like signs to c and f, and unlike to b and e.

If we suppose the reflections and refractions accompanied by changes of phase, it is easy to prove, from equations (3) and (4), that when Newton's rings are formed between two transparent media of the same kind, the intensities of the light in the reflected and transmitted systems are given by the same formulæ as when there are no changes of phase,

provided only we replace the retardation $\frac{2\pi}{\lambda} V$ (according to the notation in Airy's *Tract*) by $\frac{2\pi}{\lambda} V - 2\epsilon$, or replace D, the distance of the media, by $D - \frac{\lambda\epsilon}{2\pi \cos\beta}$.

Let us now consider some circumstances which might at first sight be conceived to affect the conclusions arrived at.

When the vibrations of the incident light take place in the plane of incidence, it appears from investigation that the conditions at the surface of separation cannot all be satisfied by means of an incident, reflected, and refracted wave, each consisting of vibrations which take place in the plane of incidence. If the media could transmit normal vibrations with velocities comparable to those with which they transmit transversal vibrations, the incident wave would occasion two reflected and two refracted waves, one of each consisting of normal, and the other of transversal vibrations, provided the angle of incidence were less than the smallest of the three critical angles (when such exist), corresponding to the refracted transversal vibrations and to the reflected and refracted normal vibrations respectively. There appear however the strongest reasons for regarding the ether as sensibly incompressible, so that the velocity of propagation of normal vibrations is incomparably greater than that of transversal vibrations. On this supposition the two critical angles for the normal vibrations vanish, so that there are no normal vibrations transmitted in the regular way whatever be the angle of incidence. Instead of such vibrations there is a sort of superficial undulation in each medium, in which the disturbance is insensible at the distance of a small multiple of λ from the surface: the expressions for these disturbances involve in fact an exponential with a negative index, which contains in its numerator the distance of the point considered from the common surface of the media. It is easy to see that the existence of the superficial undulations above mentioned does not affect the truth of equations (1), (2), (3), (4); for, to obtain these equations, it is sufficient to consider points in the media whose distances from the surface are greater than that for which the superficial undulations are sensible.

No notice has hitherto been taken of a possible motion of the material molecules, which we might conceive to be produced by the vibrations of the ether. If the vibrations of the molecules take place in the same period as those of

the ether, and if moreover they are not propagated in the body either regularly, with a velocity of propagation of their own, or in an irregular manner, the material molecules and the ether form a single vibrating system; they are in fact as good as a single medium, and the principle of reversion will apply.

In either of the excepted cases, however, the principle would not apply, for the same reason that it might lead to false results if there were normal vibrations produced as well as transversal, and the normal vibrations were not taken into account. In the case of transparent media, in which there appears to be no sensible loss of light by absorption for the small thicknesses of the media with which we are concerned in considering the laws of reflection and refraction, we are led to suppose, either that the material molecules are not sensibly influenced by the vibrations of the ether, or that they form with the ether a single vibrating system; and consequently the principle of reversion may be applied. In the case of opaque bodies, however, it seems likely that the labouring force brought by the incident luminous vibrations is partly consumed in producing an irregular motion among the molecules themselves.

When a convex lens is merely laid on a piece of glass, the central black spot is not usually seen; the centre is occupied by the colour belonging to a ring of some order. It requires the exertion of a considerable amount of pressure to bring the glasses into sufficiently intimate contact to allow of the perfect formation of the central spot.

Suppose that we deemed the glasses to be in contact when they were really separated by a certain interval Δ, and for simplicity suppose the reflections and refractions unaccompanied by any change of phase, except the loss of half an undulation. It evidently comes to the same thing to suppose the reflections and refractions to take place at the surfaces at which they do actually take place, as to suppose them to to take place at a surface midway between the glasses, and to be accompanied by certain changes of phase; and these changes ought to satisfy equations (3) and (4). This may be easily verified. In fact, putting μ, μ' for the refractive indices of the first and second media, i, i' for the angles of incidence and refraction, we easily find, by calculating the retardations, that

$$\beta = \frac{2\pi\Delta}{\lambda}\,\mu\cos i, \quad \gamma = \frac{\pi\Delta}{\lambda}\,\frac{\mu}{\sin i'}\sin(i' - i);$$

from which we get, by interchanging i and i', μ and μ', and

changing the signs, since for the first reflection and refraction the true surface comes before the supposed, but for the second the supposed surface comes before the true,

$$\varepsilon = -\frac{2\pi\Delta}{\lambda}\mu' \cos i', \quad \phi = \frac{\pi\Delta}{\lambda}\frac{\mu'}{\sin i}\sin(i' - i);$$

and these values satisfy equations (3) and (4), as was foreseen.

Hitherto the common surface of the media has been spoken of as if the media were separated by a perfectly definite surface, up to which they possessed the same properties respectively as at a distance from the surface. It may be observed, however, that the application of the principle of reversion requires no such restriction. We are at liberty to suppose the nature of the media to change in any manner in approaching the common surface; we may even suppose them to fade insensibly into each other; and these changes may take place within a distance which need not be small in comparison with λ.

It may appear to some to be superfluous to deduce particular results from hypotheses of great generality, when these results may be obtained, along with many others which equally agree with observation, from more refined theories which start with more particular hypotheses. And indeed, if the only object of theories were to group together observed facts, or even to allow us to predict the results of observation in cases not very different from those already observed, and grouped together by the theory, such a view might be correct. But theories have a higher aim than this. A well-established theory is not a mere aid to the memory, but it professes to make us acquainted with the real processes of nature in producing observed phenomena. The evidence in favour of a particular theory may become so strong that the fundamental hypotheses of the theory are hardly less certain than observed facts. The probability of the truth of the hypotheses, however, cannot be greater than the improbability that another set of equally simple hypotheses should be conceivable, which should equally well explain all the phenomena. When the hypotheses are of a general and simple character, the improbability in question may become extremely strong; but it diminishes in proportion as the hypotheses become more particular. In sifting the evidence for the truth of any set of hypotheses, it becomes of great importance to consider whether the phenomena explained, or some of them, are explicable on more simple and general hypotheses, or whether they appear absolutely to require the more particular restric-

tions adopted. To take an illustration from the case in hand, we may suppose that some theorist, starting with some particular views as to the cause of the diminished velocity of light in refracting media, and supposing that the transition from one medium to another takes place, if not abruptly, at least in a space which is very small compared with λ, has obtained as the result of his analysis M. Arago's law and the loss of half an undulation. We may conceive our theorist pointing triumphantly to these laws as an evidence of the correctness of his particular views. Yet, as we have seen, if these were the only laws obtained, the theorist would have absolutely no solid evidence of the truth of the particular hypotheses with which he started.

This fictitious example leads to the consideration of the experimental evidence for Fresnel's expressions for the intensity of reflected and refracted polarized light.

There are three particular angles of incidence, namely the polarizing angle, the angle of 90°, and the angle 0°, for which special results are deducible from Fresnel's formulæ, which admit of being put, and which have been put, to the test of experiment. The accordance of the results with theory is sometimes adduced as evidence of the truth of the formulæ: but this point will require consideration.

In the first place, it follows from Fresnel's formula for the intensity of reflected light which is polarized in a plane perpendicular to the plane of incidence, that at a certain angle of incidence the reflected light vanishes; and this angle is precisely that determined by experiment. This result is certainly very remarkable. For Fresnel's expressions are not mere empirical formulæ, chosen so as to satisfy the more remarkable results of experiment. On the contrary, they were obtained by him from dynanical considerations and analogies, which, though occasionally somewhat vague, are sufficient to lead us to regard the formulæ as having a dynamical foundation, as probably true under circumstances which without dynamical absurdity might be conceived to exist; though whether those circumstances agree with the actual state of reflecting transparent media is another question. Consequently we should *a priori* expect the formulæ to be either true or very nearly true, the difference being attributable to some modifying cause left out of consideration, or else to be altogether false: and therefore the verification of the formulæ in a remarkable, though a particular case, may be looked on as no inconsiderable evidence of their general truth. It will be observed that the truth of the formulæ is

here spoken of, not the truth of the hypotheses concerned in obtaining them from theory.

Nevertheless, even the complete establishment of the formula for the reflection of light polarized in a plane perpendicular to the plane of incidence would not establish the formula for light polarized in the plane of incidence, although it would no doubt increase the probability of its truth, inasmuch as the two formulæ were obtained in the same sort of way. But, besides this, the simplicity of the law, that the reflected ray vanishes when its direction becomes perpendicular to that of the refracted ray, is such as to lead us to regard it as not improbable that different formulæ, corresponding to different hypotheses, should agree in this point. And in fact the investigation shews that when sound is reflected at the common surface of two gases, the reflected sound vanishes when the angle of incidence becomes equal to what may be called, from the analogy of light, the polarizing angle. It is true that the formula for the intensity of the reflected sound agrees with the formula for the intensity of reflected light when the light is polarized in a plane perpendicular to the plane of incidence, and that it is the truth of the formulæ, not that of the hypotheses, which is under consideration. Nevertheless the formulæ require further confirmation.

When the angle of incidence becomes 90°, it follows from Fresnel's expressions that, whether the incident light is polarized in or perpendicularly to the plane of incidence, the intensity of the reflected light becomes equal to that of the incident, and consequently the same is true for common light. This result has been compared with experiment, and the completeness of the reflection at an incidence of 90° has been established.* The evidence, however, for the truth of Fresnel's formulæ which results from this experiment is but feeble: for the result follows in theory from the principle of *vis viva*, provided we suppose none of the labouring force brought by the incident light to be expended in producing among the molecules of the reflecting body a disturbance which is propagated into the interior, as appears to be the case with opaque bodies. Accordingly a great variety of different particular hypotheses, leading to formulæ differing from one another, and from Fresnel's, would agree in giving a perfect reflection at an incidence of 90°. Thus for example

* Transactions of the Royal Irish Academy, vol. XVII. p. 171.

the formula which Green has given* for the intensity of the reflected light, when the incident light is polarized in a plane perpendicular to the plane of incidence, gives the intensities of the incident and reflected light equal when the angle of incidence becomes 90°, although the formula in question differs from Fresnel's, with which it only agrees to a first approximation when μ is supposed not to differ much from 1. It appeared in the experiment last mentioned that the sign of the reflected vibration was in accordance with Fresnel's formulæ, and that there was no change of phase. Still it is probable that a variety of formulæ would agree in these respects.

When the angle of incidence vanishes, it follows from Fresnel's expressions, combined with the fundamental hypotheses of the theory of transversal vibrations, that if the incident light be circularly polarized, the reflected light will be also circularly polarized, but of the opposite kind, the one being right-handed, and the other left-handed.† The experiment has been performed, at least performed for a small angle of incidence,‡ from whence the result which would have been observed at an angle of incidence 0° may be inferred; and theory has proved to be in complete accordance with experiment. Yet this experiment, although confirming the theory of transversal vibrations, offers absolutely no confirmation of Fresnel's formulæ. For when the angle of incidence vanishes, there ceases to be any distinction between light polarized in, and light polarized perpendicularly to the plane of incidence: be the intensity of the reflected light what it may, it must be the same in the two cases; and this is all that it is necessary to assume in deducing the result from theory. The result would necessarily be the same in the case of metallic reflection, although Fresnel's formulæ do not apply to metals.

By the fundamental hypotheses of the theory of transverse vibrations, are here meant the suppositions, first, that the vibrations, at least in vacuum and in ordinary media, take place in the front of the wave; and secondly, that the vibrations in the case of plane polarized light are, like all the phenomena presented by such light, symmetrical with respect to the plane of polarization, and consequently are rectilinear, and take place either in, or perpendicularly to the plane of polarization. From these hypotheses, combined with the

* Transactions of the Cambridge Philosophical Society, vol. VII. p. 22.
† Philosophical Magazine, (*New Series*) vol. XXII. p. 92. ‡ *Ibid.* p. 262.

principle of the superposition of vibrations, the nature of circularly and elliptically polarized light follows. As to the two suppositions above mentioned respecting the direction of the vibrations in plane polarized light, there appears to be nothing to choose between them, so far as the geometrical part of the theory is concerned: they represent observed facts equally well. The question of the direction of the vibrations, it seems, can only be decided, if decided at all, by a dynamical theory of light. The evidence accumulated in favour of a particular dynamical theory may be conceived to become so strong as to allow us to regard as decided the question of the direction of the vibrations of plane polarized light. It appears, however, that Fresnel's expressions for the intensities, and the law which gives the velocities of plane waves in different directions within a crystal, have been deduced, if not exactly, at least as approximations to the exact result, from different dynamical theories, in some of which the vibrations are supposed to be in, and in others perpendicular to the plane of polarization.

It is worthy of remark that, whichever supposition we adopt, the direction of revolution of an ethereal particle in circularly polarized light formed in a given way is the same. Similarly, in elliptically polarized light the direction of revolution is the same on the two suppositions, but the plane which on one supposition contains the major axis of the ellipse described, on the other supposition contains the minor axis. Thus the direction of revolution may be looked on as established, even though it be considered doubtful whether the vibrations of plane polarized light are in, or perpendicular to the plane of polarization.

The verification of Fresnel's formulæ for the three particular angles of incidence above mentioned is, as we have seen, not sufficient: the formulæ however admit of a very searching comparison with experiment in an indirect way, which does not require any photometrical processes. When light, polarized in a plane making a given angle with the plane of incidence, is incident on the surface of a transparent medium, it follows from Fresnel's formulæ that both the reflected and the refracted light are plane polarized, and the azimuths of the planes of polarization are known functions of the angles of incidence and refraction, and of the azimuth of the plane of polarization of the incident light, the same formulæ being obtained whether the vibrations of plane polarized light are supposed to be in, or perpendicular to the plane of polarization. It is found by experiment that the

reflected or refracted light is plane polarized, at least if substances of a very high refractive power be excepted, and that the rotation of the plane of polarization produced by reflection or refraction agrees with the rotation determined by theory. This proves that the two formulæ, that is to say the formula for light polarized in, and for light polarized perpendicularly to the plane of incidence, are either both right, within the limits of error of very precise observations, or both wrong in the same ratio, where the ratio in question may be any function of the angles of incidence and refraction. There does not appear to be any reason for suspecting that the two formulæ for reflection are both wrong in the same ratio. As to the formulæ for refraction, the absolute value of the displacement will depend on the particular theory of refraction adopted. Perhaps it would be best, in order to be independent of any particular theory, to speak, not of the absolute displacement within a refracting medium, but of the equivalent displacement in vacuum, of which all that we are concerned to know is, that it is proportional to the absolute displacement. By the *equivalent displacement in vacuum*, is here meant the displacement which would exist if the light were to pass perpendicularly, and therefore without refraction, out of the medium into vacuum, without losing *vis viva* by reflection at the surface. It is easy to prove that Fresnel's formulæ for refraction would be adapted to this mode of estimating the vibrations by multiplying by $\sqrt{\mu}$; indeed, the formulæ for refraction might be thus proved, except as to sign, by means of the principle of *vis viva*, the formulæ for reflection being assumed. It will be sufficient to shew this in the case of light polarized in the plane of incidence.

Let i, i' be the angles of incidence and refraction, A any area taken in the front of an incident wave, l the height of a prism having A for its base and situated in the first medium. Let r be the coefficient of vibration in the reflected wave, that in the incident wave being unity, q the coefficient of the vibration in vacuum equivalent to the refracted vibration. Then the incident light which fills the volume Al will give rise to a quantity of reflected light filling an equal volume Al, and to a quantity of refracted light which, after passing into vacuum in the way supposed, would fill a volume $Al\dfrac{\cos i'}{\cos i}$. We have therefore, by the principle of *vis viva*,

$$q^2\frac{\cos i'}{\cos i} = 1 - r^2 = 1 - \frac{\sin^2(i'-i)}{\sin^2(i'+i)} = \frac{4\sin i'\cos i'\sin i\cos i}{\sin^2(i'+i)}.$$

This equation does not determine the sign of q: but it seems impossible that the vibrations due to the incident light in the ether immediately outside the refracting surface should give rise to vibrations in the opposite direction in the ether immediately inside the surface, so that we may assume q to be positive. We have then

$$q = \frac{2 \cos i \sqrt{(\sin i' \sin i)}}{\sin (i' + i)} = \sqrt{\mu} \cdot \frac{2 \sin i' \cos i}{\sin (i' + i)} \quad \ldots .(5),$$

as was to be proved. The formula for light polarized perpendicularly to the plane of incidence may be obtained in the same way. The formula (5), as might have been foreseen, applies equally well to the hypothesis that the diminished velocity of propagation within refracting media is due to an increase of density of the ether, which requires us to suppose that the vibrations of plane polarized light are perpendicular to the plane of polarization, and to the hypothesis that the diminution of the velocity of propagation is due to a diminution of elasticity, which requires us to suppose the vibrations to be in the plane of polarization.

If the refraction, instead of taking place out of vacuum into a medium, takes place out of one medium into another, it is easy to shew that we have only got to multiply by $\sqrt{\frac{\mu'}{\mu}}$ instead of $\sqrt{\mu}$, μ; μ' being the refractive indices of the first and second media respectively.

ON A GENERAL THEOREM OF DEFINITE INTEGRATION.

By George Boole. P. Math. Q.C.C.

The result which I intend to exhibit in this paper was obtained nearly two years ago. I refrained from publishing it, having designed to continue the train of investigations. I think that there is a fair promise of interesting if not of important consequences, but I have no intention now of pursuing the inquiry.

Theorem. If R be any rational fraction, a function of x, and $f(v)$ any function whatever of v, continuous or discontinuous, and if the values of x given by the equation

$$x - R = v$$

are real for all values of v included within the actual limits of integration; then, universally,

$$\int_{-\infty}^{\infty} dx\, f(x - R) = \int_{-\infty}^{\infty} dv\, f(v) \ldots\ldots\ldots (1).$$

This is the general statement of the theorem; but particular assumptions conduct us to a great variety of included cases.

First, let $f(v)$ be continuous, then the actual as well as the expressed limits of integration are $-\infty$ and ∞, and we have

$$\int_{-\infty}^{\infty} dx\, f(x - R) = \int_{-\infty}^{\infty} dv\, f(v) \ldots\ldots\ldots (2),$$

provided that the roots of the equation

$$x - R = v \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

are real for all real values of v.

Of this theorem there are many particular forms, but the following is one of the most beautiful:

$$\int_{-\infty}^{\infty} dx\, f\left(x - \frac{a_1}{x - \lambda_1} - \frac{a_2}{x - \lambda_2} \ldots - \frac{a_n}{x - \lambda_n}\right) = \int_{-\infty}^{\infty} dv\, f(v) \ldots (4),$$

provided that $a_1\ a_2 \ldots a_n$ are positive. The constants $\lambda_1\ \lambda_2 \ldots \lambda_n$ may be positive or negative, but must be real.

Suppose that $f(v)$ is discontinuous, let it be imagined to vanish for all values of v which do not lie within the limits p and q. These are then the actual limits of integration. According to this definition of the character of the function f, it is evident that $f(x - R)$ will vanish whenever $x - R$ transcends the limits p and q.

Let $p_1\ p_2 \ldots p_n$ be the roots of the equation

$$x - R = p,$$

and $q_1\ q_2 \ldots q_n$ those of the equation

$$x - R = q \ldots\ldots\ldots\ldots\ldots (6):$$

and suppose $p_1\ p_2 \ldots p_n$ and $q_1\ q_2 \ldots q_n$ arranged in the same order of magnitude, we have then

$$\int_{p_1}^{q_1} dx\, f(x-R) + \int_{p_2}^{q_2} dx\, f(x-R) \ldots + \int_{p_n}^{q_n} dx\, f(x-R) = \int_{p}^{q} dv\, f(v) \ldots (7),$$

and this may be applied to the determination of the sums of an infinite variety of transcendental integrals.

We here see an unexpected connexion established between the theory of single definite integrals and that of the sums of

integrals. The latter question, as exhibited in the works of Abel and others, does not appear to suggest the existence of any such relation.

I proceed to offer one or two verifications, and I begin with the very simple formula deduced from (4),

$$\int_{-\infty}^{\infty} dx f\left(x - \frac{a}{x}\right) = \int_{-\infty}^{\infty} dv f(v) \ldots\ldots\ldots (8).$$

Let $f(v) = \varepsilon^{-v^2}$, then $f\left(x - \frac{a}{x}\right) = \varepsilon^{-\left(x-\frac{a}{x}\right)^2}$,

whence, $\int_{-\infty}^{\infty} dx \varepsilon^{-\left(x^2-2a+\frac{a^2}{x^2}\right)} = \int_{-\infty}^{\infty} dv \varepsilon^{-v^2} = \pi^{\frac{1}{2}}$;

therefore $\varepsilon^{2a} \int_{-\infty}^{\infty} dx \varepsilon^{-\left(x^2+\frac{a^2}{x^2}\right)} = \pi^{\frac{1}{2}}$,

$$\int_{-\infty}^{\infty} dx \varepsilon^{-\left(x^2+\frac{a^2}{x^2}\right)} = \pi^{\frac{1}{2}}\varepsilon^{-2a},$$

$$\int_{0}^{\infty} dx \varepsilon^{-\left(x^2+\frac{a^2}{x^2}\right)} = \tfrac{1}{2}\pi^{\frac{1}{2}}\varepsilon^{-2a} \ldots\ldots\ldots (9),$$

which is well known; and in an exactly similar manner we may deduce

$$\int_{0}^{\infty} dx \cos\left(x^2 + \frac{a^2}{x^2}\right) \int_{0}^{\infty} dx \sin\left(x^2 + \frac{a^2}{x^2}\right).$$

If n be even, we have

$$\int_{-\infty}^{\infty} dv \varepsilon^{-v^n} = \frac{2}{n}\Gamma\left(\frac{1}{n}\right) \ldots\ldots\ldots (10).$$

Hence, $\int_{-\infty}^{\infty} dx \varepsilon^{-\left(x-\frac{a}{x}\right)^n} = \frac{2}{n}\Gamma\left(\frac{1}{n}\right)$,

which is a more general form. Let $n = 4$, and we have

$$\int_{-\infty}^{\infty} dx\, \varepsilon^{-\left(x^4+\frac{a^4}{x^4}\right)+4a\left(x^2+\frac{a^2}{x^2}\right)} = \frac{2}{n}\Gamma\left(\frac{1}{n}\right)\varepsilon^{6a^2},$$

therefore $\int_{0}^{\infty} dx \varepsilon^{-\left(x^4+\frac{a^4}{x^4}\right)+4a\left(x^2+\frac{a^2}{x^2}\right)} = \frac{1}{n}\Gamma\left(\frac{1}{n}\right)\varepsilon^{6a^2} \ldots (11)$;

and similarly for other values of n. Perhaps this is new.

From the known integral $\int_0^\infty \frac{d\theta\, \theta^{-\frac{1}{2}}}{(1+\theta)^n}$, I in like manner deduced

$$\left.\begin{aligned}\int_0^\infty \frac{dx\, x^{n-\frac{3}{2}}}{(a+bx+cx^2)^n} &= \frac{\Gamma(n-\frac{1}{2})\pi^{\frac{1}{2}}}{\Gamma(n)\, a^{\frac{1}{2}}}\, \frac{1}{(b+2\sqrt{ac})^{n-\frac{1}{2}}}\\ \int_0^\infty \frac{dx\, x^{n-\frac{1}{2}}}{(a+bx+cx^2)^n} &= \frac{\Gamma(n-\frac{1}{2})\pi^{\frac{1}{2}}}{\Gamma(n)\, c^{\frac{1}{2}}}\, \frac{1}{(b+2\sqrt{ac})^{n-\frac{1}{2}}}\end{aligned}\right\}\ldots(12);$$

which agree with formulæ published by Professor Thomson (vol. II. p. 122) and Mr. Cayley (vol. II. p. 125).

Generally however the results of this method do not appear to be attainable by any other, and must therefore rest on their own evidence.

Thus, from the known definite integral

$$\int_{-\infty}^\infty \frac{dx \cos ax}{1+x^2} = \pi\varepsilon^{-a},$$

we have, supposing r positive,

$$\int_{-\infty}^\infty \frac{dx \cos\left(ax - \frac{ar}{x}\right)}{1+\left(x-\frac{r}{x}\right)^2} = \pi\varepsilon^{-a};$$

and an infinite number of similar results, but their verification is not easy.

In connexion with (8) the properties of the definite integral

$$\int_{-\infty}^\infty dx f\left(x+\frac{a}{x}\right)$$

seem worth investigating. It is obvious that x becomes imaginary in the equation

$$x+\frac{a}{x}=v,$$

whenever v lies, between the limits $-2\sqrt{a}$ and $2\sqrt{a}$. The integral with reference to v consists therefore of two real portions; in one of which v varies from $2\sqrt{a}$ to ∞, in the other from $-\infty$ to $-2\sqrt{a}$.

Consider the former. Now

$$x+\frac{a}{x}=2\sqrt{a} \text{ gives } x=\sqrt{a} \text{ (roots equal)},$$

$$x+\frac{a}{x}=\infty \text{ gives } x=0 \text{ or } \infty.$$

Hence by (7) $\left(\int_{\sqrt{a}}^{0} + \int_{\sqrt{a}}^{\infty}\right) dx f\left(x + \frac{a}{x}\right) = \int_{2\sqrt{a}}^{\infty} dv f(v),$

or $$\int_0^{\infty} dx f\left(x + \frac{a}{x}\right) - 2\int_0^{\sqrt{a}} dx f\left(x + \frac{a}{x}\right) = \int_{2\sqrt{a}}^{\infty} dv f(v) \ldots (13),$$

which is true for all forms of f. Similarly

$$\int_{-\infty}^{0} dx f\left(x + \frac{a}{x}\right) - 2\int_{-\sqrt{a}}^{0} dx f\left(x + \frac{a}{x}\right) = \int_{-\infty}^{-2\sqrt{a}} dv f(v) \ldots (14);$$

if we add (13) and (14) we get on reduction

$$\int_{-\infty}^{\infty} dx f\left(x + \frac{a}{x}\right) - 2\int_{-\sqrt{a}}^{\sqrt{a}} dx f\left(x + \frac{a}{x}\right) = \int_{-\infty}^{\infty} dv f(v) - \int_{-2\sqrt{a}}^{2\sqrt{a}} dv f(v) \ldots (15).$$

From these some very curious relations may be deduced. In illustration of (13) let $f(v) = \varepsilon^{2a-v^2}$ then,

$$\int_0^{\infty} dx \varepsilon^{-\left(x^2 + \frac{a^2}{x^2}\right)} - 2\int_0^{\sqrt{a}} dx \varepsilon^{-\left(x^2 + \frac{a^2}{x^2}\right)} = \int_{2\sqrt{a}}^{\infty} dv \varepsilon^{2a-v^2},$$

$$= \varepsilon^{2a}\left(\int_0^{\infty} dv \varepsilon^{-v^2} - \int_0^{2\sqrt{a}} dv \varepsilon^{-v^2}\right).$$

Substituting for the first term in each member its known value, we have

$$\frac{\pi^{\frac{1}{2}}}{2}\varepsilon^{-2a} - 2\int_0^{\sqrt{a}} dx \varepsilon^{-\left(x^2 + \frac{a^2}{x^2}\right)} = \varepsilon^{2a}\frac{\pi^{\frac{1}{2}}}{2} - \varepsilon^{2a}\int_0^{2\sqrt{a}} dv \varepsilon^{-v^2};$$

$$\therefore \int_0^{\sqrt{a}} dx \varepsilon^{-\left(x^2 + \frac{a^2}{x^2}\right)} = \frac{\varepsilon^{2a}}{2}\int_0^{2\sqrt{a}} dv \varepsilon^{-v^2} - \frac{\pi^{\frac{1}{2}}}{4}\left(\varepsilon^{2a} - \varepsilon^{-2a}\right) \ldots (16),$$

and such relations may be indefinitely multiplied. By reversion of (8) the reader will easily prove the following remarkable theorem,

$$\int_{-\infty}^{\infty} dv f\{v \pm \sqrt{(v^2 + 4a)}\} = \int_0^{\infty} dx f(\pm 2x) + \int_0^{\infty} dx f\left(\pm \frac{2a}{x}\right) \ldots (17),$$

and the two particular theorems included in this form may easily be shewn to verify each other.

From a great number of other results which I have noted, I select the following (mn positive),

$$\left(\int_{-m}^{-n} + \int_{n}^{m}\right) dx f\left(x - \frac{mn}{x}\right) = \int_{-(m-n)}^{m-n} dv f(v). \ldots (18),$$

whence, if $f(v)$ is an even function,

$$\int_n^m dx\, f\left(x - \frac{mn}{x}\right) = \int_0^{m-n} dv\, f(v);$$

let $x = t + n$, then

$$\int_0^{m-n} dt\, f\left(t + n - \frac{mn}{t+n}\right) = \int_0^{m-n} dv\, f(v).$$

Let $m - n = r$, then we have

$$\int_0^r dv\, f(v) = \int_0^r dt\, f\left\{t + n - \frac{n(n+r)}{n+t}\right\} \ldots\ldots (19),$$

provided that $n(n + r)$ is positive.

And generally, *any definite integral of an even function being given within assigned limits, we can deduce the values of an infinite number of other integrals within the same limits.* I have in some instances verified results thus obtained, but usually the integration appears impossible by other means.

To the above general conclusion we may add the following. *A definite integral of any odd function being given within assigned limits, we can determine an infinite number of sums of transcendental integrals, the number of integrals varying in each case from two upwards, ad libitum.*

A definite integral of any function, odd or even, being given within the limits $-\infty$ *and* ∞, *we can deduce from it the values of an infinite number of single definite integrals within the same limits.*

In all applications of these theorems the imaginary term, which appears when we integrate across an infinite element, must be rejected according to Cauchy's rule.

The reduction of definite multiple integrals usually depends on simple integrations of the form

$$\int_{-\infty}^{\infty} dv \cos\{\phi(v)\};$$

this is known to be integrable when $\phi(v)$ is of the form $a + bv + cv^2$. The general theorem of this paper shews that it is also integrable by reduction to the same form in an infinite number of other cases. Of these I have examined only a few. From the results, I am led to anticipate that the most general expression for the value of the multiple integral

$$V = \iint \ldots dx_1\, dx_2 \ldots dx_n \frac{f\{\phi_1(x_1) + \phi_2(x_2) \ldots + \phi_n(x_n)\}}{\{\psi_1(x_1) + \psi_2(x_2) \ldots + \psi_n(x_n)\}^i},$$

subject to the condition

$$\phi_1(x_1) + \phi_2(x_2) \ldots + \phi_n(x_n) \overline{\overline{<}}\ 1,$$

is usually of the following form:

$$V = \frac{\pi^{\frac{1}{2}n}}{\Gamma(i)} \int_0^\infty ds \frac{s^{i-1}}{\phi^{\frac{1}{2}}} \left(-\frac{d}{d\sigma}\right)^{i-\frac{1}{2}n} f(\sigma),$$

in which σ is a function of s in the form of a *sum*, ϕ a function of s in the form of a *product*, and $f(\sigma)$ a discontinuous function of σ which vanishes whenever σ lies without the limits 0 and 1. All the forms for such multiple integrals with which I am acquainted really belong to this, and it would not, I conceive, be difficult to assign the general values of ϕ and 0.

One of the most general cases which I have been able to discuss is the following:

$$V = \iint dx_1\, dx_2 \ldots \frac{f(h_1^2 X_1^2 + h_2^2 X_2^2 \ldots + h_n^2 X_n^2)}{\{h^2 + (a_1 - X_1)^2 + (a_2 - X_2)^2 \ldots + (a_n - X_n)^2\}^i},$$

subject to the condition

$$h_1^2 X_1^2 + h_2^2 X_2^2 \ldots + h_n^2 X_n^2 \overline{\overline{<}}\ 1,$$

in which $X_1 = x_1 - R_1 \quad X_2 = x_2 - R_2 \ldots X_n = x_n - R_n$,

when $R_1\ R_2 \ldots R_n$ are any rational functions of $x_1\ x_2 \ldots x_n$ respectively subject to the several conditions that the equations

$$x_1 - R_1 = v \quad x_2 - R_2 = v \ldots x_n - R_n = v,$$

shall have none but real values of $x_1\ x_2 \ldots x_n$ for all real values of v. It is not necessary that $R_1\ R_2 \ldots R_n$ should be similar functions of the respective variables.

I am not without a hope that the extreme generality of the theorem, of which some illustrations have been given in this paper, and the great facility and variety of its applications, will induce others to examine it more fully than I have been able to do.

Lincoln, Dec. 24, 1847.

ON DIFFERENTIATION WITH FRACTIONAL INDICES, AND ON GENERAL DIFFERENTIATION.

II.—ON GENERAL DIFFERENTIATION.

By the Rev. WILLIAM CENTER, A.M.

[*Continued from* Vol. III. p. 275.]

§ 1. *Investigation of Formulæ of Reduction.*

THE formulæ of general differentiation, (a), (b), (c), (d), (e), and (f), already given, and deduced directly from the fundamental definite integral

$$\left(\frac{d}{dx}\right)^\theta \frac{1}{x^m} = \frac{(-1)^\theta}{\Gamma m} \int_0^\infty \varepsilon^{-ax} a^{m+\theta-1} da = (-1)^\theta \frac{\Gamma(m+\theta)}{\Gamma m} \cdot \frac{1}{x^{m+\theta}},$$

by a process altogether within the proper limit of Legendre's function, comprehend the whole theory of general differentiation. It would be convenient, however, and on that account desirable, could we reduce these to one *universal* formula, that should just express them and nothing more. It is the object of the present paper to shew how this may be accomplished.

Although Legendre's function Γp virtually ceases to exist at the exact limit $p = 0$, at which, $\Gamma 0 =$ infinity; yet a mere extension of the Γ notation, under the proper prescription of meaning affixed to it, may prove very serviceable. Thus, when p is a proper fraction (positive), we may be allowed to form the conventional equality $\Gamma(1-p) = -p\,\Gamma(-p)$, provided we employ the second member $(-p\,\Gamma(-p))$ merely as an algebraical equivalent for the first $\Gamma(1-p)$, and affix to it no meaning beyond. In such a case, the *negative* forms would receive their interpretation from their *positive* equivalents that lie within the proper limits of Legendre's function. In the same way, if r be any number less than 2, we would have, by a similar extension of the Γ notation, $\Gamma(2-r) = (1-r)\,\Gamma(1-r)$, where the second member is interpretable only by the first. We can thus obtain an extension of our notation without any sacrifice of logical accuracy; for in strict analytical truth, there is no *real* Γ function beyond the exact limit $\Gamma 0$, but only a functional *notation*.

Granting this convention, the only thing essential to the introduction of it into rigorous demonstration is, that our starting-point must invariably lie within the proper limit of Legendre's function. When from this starting-point we

have made the *descent*, then only can we with certainty retrace our steps in the *ascent*. In short, after we have passed in a direct and continuous line from the fundamental definite integral

$$\left(\frac{d}{dx}\right)^{\theta} x^{-m} = \frac{(-1)^{\theta}}{\Gamma(m)} \int_0^{\infty} \varepsilon^{-ax} a^{m+\theta-1} da,$$

to the negative form of Liouville's formula,

$$\left(\frac{d}{dx}\right)^{\theta} x^{m} = (-1)^{\theta} \frac{\Gamma(-m+\theta)}{\Gamma(-m)} x^{m-\theta},$$

can we be said to have demonstrated its generality. The present problem of reduction, – though not *necessary* to the theory of general differentiation, – is of considerable analytical importance, as tending to throw some additional light upon this perplexing subject. It will shew the precise sense in which we are to understand Liouville's formula to be *general*.

(1) Adopting then the proposed extension of the Γ notation, subject to the conditions before prescribed, let p be any positive proper fraction, then

$$\Gamma(1-p) = -p\,\Gamma(-p),$$

$$= (-p)(-p-1)(-p-2)\ldots.(-p-n)\,\Gamma(-p-n)$$

$$= (-1)^{n+1}.p.(p+1)(p+2)\ldots.(p+n)\,\Gamma(-p-n)$$

$$\therefore\ \Gamma(1-p) = (-1)^{n+1} p \frac{\Gamma(1+p+n)}{\Gamma(1+p)} \Gamma(-p-n)\ldots\ldots\ldots\ldots(1).$$

It is proper here to remark, that n is *any* positive whole number. Again, let $p = 1 - q$, where p and q are both proper fractions; then we may perform a similar operation upon $\Gamma(1+p)$; for then

$$\Gamma(1+p) = p\,\Gamma(p) = p\,\Gamma(1-q),$$

$$\text{by (1)}, \quad = (-1)^{n+1} \frac{p.q.\Gamma(1+q+n)}{\Gamma(1+q)} \Gamma(-q-n)$$

$$= (-1)^{n+1} \frac{p\,\Gamma(1+q+n)}{\Gamma(q)} \Gamma(-q-n),$$

$$\text{or,} \quad \Gamma(1+p) = (-1)^{n+1} \frac{(-p)\,\Gamma(q+n)\,(-q-n)\,\Gamma(-q-n)}{\Gamma(q)}$$

$$= (-1)^{n+1} \frac{(-p)\,\Gamma(q+n)}{\Gamma(q)} \Gamma(1-q-n);$$

and since $p = 1 - q$, therefore

$$\Gamma(1+p) = (-1)^{n+1}\frac{(-p)\,\Gamma(1-p+n)}{\Gamma(1-p)}\,\Gamma(p-n) \;\ldots\ldots (2).$$

On comparing (1) and (2) we immediately perceive that the one passes into the other by merely changing the sign of p; so that we may enunciate the relation generally, and say, that if p be any proper fraction either positive or negative, while n is *any* positive whole number, the following conventional equality subsists:

$$\Gamma(1-p) = (-1)^{n+1}\frac{p.\Gamma(1+p+n)}{\Gamma(1+p)}\,\Gamma(-p-n) \;\ldots\ldots (\mathrm{F}).$$

(2) Let us now recur to our general differential notation β, λ, μ, and m, n, p, as already defined; and since $m = n + p$, we have by substitution in (F)

$$\Gamma(1-p) = (-1)^{n+1}\frac{p.\Gamma(1+m)}{\Gamma(1+p)}\,\Gamma(-m) \ldots\ldots\ldots (3).$$

(3) Since p and μ are both proper fractions, then $(p - \mu)$ is also a proper fraction either positive or negative. Also, when $\beta < n$, then $(n - \beta)$ is a positive whole number, since β and n are both positive whole numbers. Let us now substitute in (F) $(p - \mu)$ for p, and $(n - \beta)$ for n, remembering that $\lambda = \beta + \mu$; hence we have

$$\Gamma(1-p+\mu) = (-1)^{n-\beta+1}.(p-\mu)\frac{\Gamma(1+p-\mu+n-\beta)}{\Gamma(1+p-\mu)}\,\Gamma(-p+\mu-n+\beta);$$

$$\therefore\; \Gamma(1-p+\mu) = (-1)^{n-\beta+1}\frac{(p-\mu)\,\Gamma(1+m-\lambda)}{\Gamma(1+p-\mu)}\,\Gamma(-m+\lambda) \ldots (4).$$

(4) Let $\beta > n$, then $(\beta - n)$ is a positive whole number; also $(\mu - p)$ is a proper fraction either positive or negative. Let us again substitute in (F), $(\mu - p)$ for p, and $(\beta - n)$ for n, when

$$\Gamma(1+p-\mu) = (-1)^{\beta-n+1}\frac{(\mu-p)\,\Gamma(1-p+\mu+\beta-n)}{\Gamma(1-p+\mu)}\,\Gamma(p-\mu-\beta+n),$$

or $$\Gamma(1+p-\mu) = (-1)^{\beta-n}\frac{(p-\mu).\Gamma(1-m+\lambda)}{\Gamma(1-p+\mu)}\,\Gamma(m-\lambda) \ldots (5).$$

(5) We yet require the aid of another general formula of reduction, which may be thus obtained. Let r be any

number less than 2; then $\Gamma(2-r)$ is within the proper limit of Legendre's function; and by our extension of the Γ notation, we have

$$\Gamma(2-r) = (1-r)\,\Gamma(1-r)$$

$$= (-1)^{n+1}.r.(1-r)\,\frac{\Gamma(1+r+n)}{\Gamma(1+r)}\,\Gamma(-r-n)\,..(\mathrm{G}).$$

Here n is *any* positive whole number, as before. Since p and μ are both proper fractions, then $(p+\mu) < 2$; also $(\beta + n)$ is a positive whole number; hence substituting in (G) $(p+\mu)$ for r, and $(\beta + n)$ for n, we have

$$\Gamma(2-p-\mu)$$

$$= (-1)^{\beta+n+1}.\,\frac{(p+\mu)(1-p-\mu)\,\Gamma(1+p+\mu+n+\beta)}{\Gamma(1+p+\mu)}\,\Gamma(-p-\mu-\beta-n),$$

or $$\Gamma(2-p-\mu) = (-1)^{\beta+n+1}.\,\frac{(p+\mu)(1-p-\mu)\,\Gamma(1+m+\lambda)}{\Gamma(1+p+\mu)}\,\Gamma(-m-\lambda)...(6).$$

§ 2. *Reduction of* (*a*), (*b*), (*c*), (*d*), (*e*), *and* (*f*).

(1) Our first formula (*a*) for general differentiation corresponds to the case of $\beta < n$, and is

$$\left(\frac{d}{dx}\right)^{\lambda} x^m = \frac{(-1)^{\mu}\,p\,\Gamma(1-p+\mu)\,\Gamma(1+p-\mu)\,\Gamma(1+m)}{(p-\mu)\,\Gamma(1-p)\,\Gamma(1+p)\,\Gamma(1+m-\lambda)}\,x^{m-\lambda}.$$

Let us substitute for $\Gamma(1-p)$ by means of (3), therefore

$$\left(\frac{d}{dx}\right)^{\lambda} x^m = \frac{(-1)^{\mu}\,\Gamma(1-p+\mu)\,\Gamma(1+p-\mu)}{(-1)^{n+1}.(p-\mu)\,\Gamma(1+m-\lambda)\,\Gamma(-m)}\,.x^{m-\lambda}.$$

Again, let us substitute in this last equation for $\Gamma(1-p+\mu)$ by (4), when, expunging the common factors, we have

$$\left(\frac{d}{dx}\right)^{\lambda} x^m = (-1)^{\mu-\beta}\,\frac{\Gamma(-m+\lambda)}{\Gamma(-m)}\,x^{m-\lambda},$$

or $$\left(\frac{d}{dx}\right)^{\lambda} x^m = (-1)^{\lambda}\,\frac{\Gamma(-m+\lambda)}{\Gamma(-m)}\,x^{m-\lambda}\,.......(a');$$

for since β is integer, $(-1)^{-\beta} = (-1)^{\beta}$; and $(-1)^{\mu-\beta} = (-1)^{\mu+\beta} = (-1)^{\lambda}$.

(2) Our second formula (b) corresponds to the case of $\beta > n$, and is

$$\left(\frac{d}{dx}\right)^{\lambda} x^m = (-1)^{\lambda - n} \frac{p \,\Gamma(1+m)\, \Gamma(1-m+\lambda)}{(m-\lambda)\, \Gamma(1-p)\, \Gamma(1+p)} x^{m-\lambda}.$$

Substitute for $\Gamma(1-p)$ by (3), as before; then

$$\left(\frac{d}{dx}\right)^{\lambda} x^m = \frac{(-1)^{\lambda - n}\, \Gamma(1-m+\lambda)}{(-1)^{n+1}(m-\lambda)\, \Gamma(-m)} x^{m-\lambda}$$

$$= \frac{(-1)^{\lambda}\, \Gamma(1-m+\lambda)}{(-1)^{2n}(-m+\lambda)\, \Gamma(-m)} x^{m-\lambda};$$

and since $(-1)^{2n} = 1$, therefore

$$\left(\frac{d}{dx}\right)^{\lambda} x^m = (-1)^{\lambda} \frac{\Gamma(-m+\lambda)}{\Gamma(-m)} x^{m-\lambda} \quad \ldots\ldots\ldots\ldots (b').$$

(3) Our third formula (c) is general for all values of β, and is

$$\left(\frac{d}{dx}\right)^{-\lambda} x^m = (-1)^{-\mu} \frac{p\, \Gamma(2-p-\mu)\, \Gamma(1+p+\mu)\, \Gamma(1+m)}{(p+\mu)(1-p-\mu)\, \Gamma(1-p)\, \Gamma(1+p)\, \Gamma(1+m+\lambda)} x^{m+\lambda}.$$

Substituting in this for $\Gamma(1-p)$ by (3), as before, therefore

$$\left(\frac{d}{dx}\right)^{-\lambda} x^m = \frac{(-1)^{-\mu}\, \Gamma(2-p-\mu)\, \Gamma(1+p+\mu)}{(-1)^{n+1}(p+\mu)(1-p-\mu)\, \Gamma(1+m+\lambda)\, \Gamma(-m)} x^{m+\lambda}.$$

Substituting now for $\Gamma(2-p-\mu)$ by (6), expunging the common factors, and remembering that we have

$$(-1)^{\beta-\mu} = (-1)^{-\beta-\mu} = (-1)^{-\lambda}, \text{ then}$$

$$\left(\frac{d}{dx}\right)^{-\lambda} x^m = (-1)^{-\lambda} \frac{\Gamma(-m-\lambda)}{\Gamma(-m)} x^{m+\lambda} \quad \ldots\ldots\ldots\ldots (c').$$

(4) Our fourth formula (d), which is

$$\left(\frac{d}{dx}\right)^{\lambda} x^{-m} = (-1)^{\lambda} \frac{\Gamma(m+\lambda)}{\Gamma(m)} x^{-m-\lambda} \quad \ldots\ldots\ldots\ldots (d'),$$

requires no reduction.

Our fifth (e) corresponds to the case of $\beta < n$, and is

$$\left(\frac{d}{dx}\right)^{-\lambda} x^{-m} = (-1)^{-\lambda} \frac{\Gamma(1+m-\lambda)}{(m-\lambda)\, \Gamma(m)} x^{-m+\lambda};$$

$$\therefore \left(\frac{d}{dx}\right)^{-\lambda} x^{-m} = (-1)^{-\lambda} \frac{\Gamma m - \lambda}{\Gamma m} x^{-m+\lambda} \quad \ldots\ldots\ldots\ldots (e').$$

(5) Our last formula (f) corresponds to the case of $\beta > n$, and is

$$\left(\frac{d}{dx}\right)^{-\lambda} x^{-m} = \frac{(-1)^{n-\mu}\, \Gamma 1 - p + \mu\, \Gamma 1 + p - \mu}{(p-\mu)\, \Gamma m\, \Gamma 1 - m + \lambda} x^{-m+\lambda}.$$

Substituting for $\Gamma 1 + p - \mu$ by (5), we have, since $(-1)^{\beta} = (-1)^{-\beta}$,

$$\left(\frac{d}{dx}\right)^{-\lambda} x^{-m} = (-1)^{-\lambda} \frac{\Gamma m - \lambda}{\Gamma m} x^{-m+\lambda} \quad \ldots\ldots (f').$$

On comparing the reduced formulæ (a'), (b'), (c'), (d'), (e'), and (f'), which comprehend all the cases of the problem of general differentiation, what is the inference that we ought to draw? Is it not this, that Liouville's formula is universal, having the same evidence for its truth as the fundamental definite integral on which its basis rests, and subject in its processes to the conventional equality $\Gamma 1 - p = -p\,\Gamma{-p}$, on which the reductions just given are founded.

Longside, Mintlaw, February 12, 1848.

ON THE THEOREMS IN SPACE ANALOGOUS TO THOSE OF PASCAL AND BRIANCHON IN A PLANE.

By Thomas Weddle.

Surfaces of the second degree are of three kinds:

(1) *Umbilical** surfaces, including the ellipsoid, the hyperboloid of two sheets, and the elliptic paraboloid:

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* This term, which seems an excellent one, was suggested to me by Mr. Hearn, of the Royal Military College. I believe it has not hitherto been employed to *classify* surfaces of the second degree.

The properties given in this paper were communicated by me to Mr. Hearn, and he has confirmed the principal results at which I have arrived, relative to umbilical surfaces, in a very simple and ingenious manner. I am also indebted to this talented mathematician for many valuable remarks, and I regret exceedingly that, consistently with the plan of this paper, I cannot make greater use of the matter which he has communicated to me. I trust however that he will hereafter publish his investigations in some shape or other.

(2) *Rule* surfaces, including the hyperboloid of one sheet and the hyperbolic paraboloid:

(3) *Developable* surfaces, including the cone and the elliptic, hyperbolic, and parabolic cylinders.

And it will, in the following somewhat elaborate discussion, be necessary to consider these three classes of surfaces separately. I begin with the umbilical surfaces, which are here very much more interesting than the others.

Let the edges of $\left\{\begin{matrix}\textit{a hexahedron}\\ \textit{an octahedron}\end{matrix}\right\}$* *touch an umbilical surface of the second degree, then*

I. *The opposite faces intersect in* $\left\{\begin{matrix}\textit{three}\\ \textit{four}\end{matrix}\right\}$ *straight lines in one plane.*

II. *The* $\left\{\begin{matrix}\textit{four}\\ \textit{three}\end{matrix}\right\}$ *straight lines joining the opposite angular points intersect in a point, and the solid has* $\left\{\begin{matrix}\textit{six}\\ \textit{three}\end{matrix}\right\}$ *diagonal planes.*

III. *The* $\left\{\begin{matrix}\textit{four}\\ \textit{three}\end{matrix}\right\}$ *diagonal straight lines, and the six straight lines joining the points of contact of opposite edges, intersect in a point.*

It may be observed in passing, that surfaces touching the edges of polyhedra seem hitherto scarcely to have been alluded to by mathematicians, and yet many most elegant properties belong to surfaces and solids so related. I would strongly recommend any one who may be in search of analogues in space to properties of plane curves, not to overlook surfaces touching the edges of polyhedra, for it seems highly probable that numerous plane properties, especially of curves of the second degree, will be found to have analogues in three dimensions relative to such surfaces and solids. At any rate it will, I hope, be shewn (partially at least) in this paper that surfaces of the second degree touching the edges of polyhedra have very interesting properties, and this will be made further evident in another paper, which will contain a large number of properties chiefly relating to surfaces of the second degree touching the edges of the tetrahedron.

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* The hexahedron and octahedron referred to may be constructed as follows: For a hexahedron, take three pairs of planes, and make each plane intersect all the others except that with which it forms a pair. Also, to construct an octahedron, take three pairs of points, and draw straight lines from each point to all the others except that with which it forms a pair; these straight lines will be the edges of the octahedron, aud the eight triangles which they form, will be its faces.

It is evident that (III.) includes (II.), but I prefer giving the enunciations as above, because I regard (I.) and (II.) as being analogous to Pascal's and Brianchon's well-known theorems, while I consider (III.) to be analogous to the following. If a quadrilateral circumscribe a conic section, the diagonals of the quadrilateral and the straight lines joining the points of contact of opposite sides intersect in a point.

Let $t = 0 \ldots\ldots (1)$, $u = 0 \ldots\ldots (2)$, $v = 0 \ldots\ldots (3)$,

be the equations to three contiguous faces of a hexahedron, then will

$$s^2 = 4\lambda uv + 4\mu tv + 4\nu tu \ldots\ldots\ldots\ldots (4)$$

be the equation to any surface of the second degree touching the mutual intersections of (1), (2), and (3), $s = 0$ being the equation to the plane which passes through the three points of contact. Supposing t, u, and v to have been at first multiplied by constants, we may denote the faces of the hexahedron that are opposite to (1), (2), and (3), respectively by

$$t + b_1 u + c_1 v = s \ldots\ldots\ldots\ldots (5),$$

$$a_2 t + u + c_2 v = s \ldots\ldots\ldots\ldots (6),$$

$$a_3 t + b_3 u + v = s \ldots\ldots\ldots\ldots (7).$$

Eliminate s and v from (4) by means of (3) and (5), therefore

$$t^2 + b_1^2 u^2 + 2(b_1 - 2\nu)\, tu = 0:$$

now, since the intersection of (3) and (5) is to touch the surface, the last equation must be a complete square; hence either $\nu = 0$ or $b_1 = \nu$, but the former must be rejected, for then (4) would denote a developable surface, consequently $b_1 = \nu$. In like manner, by considering that each of the lines (3, 5), (3, 6), (2, 5), (2, 7), (1, 6), and (1, 7), must touch the surface, we have

$$b_1 = a_2 = \nu,$$

$$c_1 = a_3 = \mu,$$

and $$c_2 = b_3 = \lambda.$$

Hence the equations (5, 6, 7) become

$$t + \nu u + \mu v = s \ldots\ldots\ldots\ldots (8),$$

$$\nu t + u + \lambda v = s \ldots\ldots\ldots\ldots (9),$$

$$\mu t + \lambda u + v = s \ldots\ldots\ldots\ldots (10),$$

and we have to determine λ, μ, and ν, so that the mutual intersections of (8), (9), and (10) may touch the surface (4).

Multiply (8) by t and (9) by u, add four times the sum of the products to (4), and reduce

$$\therefore (s - 2t - 2u)^2 = - 4 (\nu - 2) tu \dots\dots\dots (11),$$

eliminate v from (8) and (9), hence

$$s = \frac{(\mu\nu - \lambda) t - (\lambda\nu - \mu) u}{\mu - \lambda},$$

and this reduces the last equation to

$$[\{(\nu - 2)\mu + \lambda\} t - \{(\nu - 2)\lambda + \mu\} u]^2 = - 4 (\lambda - \mu)^2 (\nu - 2) tu.$$

Since the straight line (8, 9) touches the surface (4), the roots of this quadratic $\left(\text{in } \dfrac{t}{u}\right)$ must be equal: and it is easily shewn that if the roots of the equation $(mx - n)^2 = - 4px$ be equal, we must have either $p = 0$, or $p = mn$; hence either

$$\nu - 2 = 0,^*$$

or $$(\lambda - \mu)^2 (\nu - 2) = \{(\nu - 2)\mu + \lambda\} \{(\nu - 2)\lambda + \mu\}$$
$$= \lambda\mu (\nu - 2)^2 + (\lambda^2 + \mu^2)(\nu - 2) + \lambda\mu.$$

Transpose, reduce, and extract the square root, $\therefore \nu = 1$,

$$\therefore \nu = 1 \text{ or } 2,$$

similarly $$\mu = 1 \text{ or } 2,$$

and $$\lambda = 1 \text{ or } 2.$$

Hence we must have

either $$\lambda = \mu = \nu = 1 \dots\dots\dots\dots (12),$$

or $$\lambda = \mu = \nu = 2 \dots\dots\dots\dots (13),$$

or, such as, $$\lambda = \mu = 2,\ \nu = 1 \dots\dots\dots (14),$$

or, such as, $$\lambda = \mu = 1,\ \nu = 2 \dots\dots\dots (15).$$

Now it will readily appear from (8, 9, 10) that (12) and (14) are inadmissible, so also is (15), for then (1), (9) and (10) would intersect in a straight line; we must therefore have $\lambda = \mu = \nu = 2$. For the sake of symmetry and simplicity, put $t = T + S$, $u = U + S$, $v = V + S$, and $s = 2T + 2U + 2V + 4S$ (assumptions which do not interfere with the generality of

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* The condition $p = 0$, that is $(\lambda - \mu)^2 (\nu - 2) = 0$, is also satisfied by $\lambda = \mu$; but in this case (8) and (9) give $t = u$, and (11) reduces to $(s - 4t)^2 = - 4 (\nu - 2) t^2$, the roots of which can be equal only when $\nu - 2 = 0$.

the equations); then (1, 2, 3, 8, 9, 10) reduce, by means of (13), to

$$T + S = 0\ldots\ldots(16), \qquad T - S = 0\ldots\ldots(17),$$
$$U + S = 0\ldots\ldots(18), \qquad U - S = 0\ldots\ldots(19),$$
$$V + S = 0\ldots\ldots(20), \qquad V - S = 0\ldots\ldots(21).$$

These are the equations to the faces of the hexahedron, the opposite faces being (16) and (17), (18) and (19), (20) and (21); and it is worthy of notice that these become the equations to the faces of a parallelepiped when S is a constant.

The equation (4) to the surface takes the very simple form

$$T^2 + U^2 + V^2 = 2S^2\ldots\ldots\ldots\ldots(22),$$

and it is easy to shew that this denotes umbilical surfaces only.

In the first place, all the umbilical surfaces are included in this equation; this is evident, for the equations to the ellipsoid, hyperboloid of two sheets, and elliptic paraboloid, may be written

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 2\left(\frac{1}{\sqrt{2}}\right)^2,$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + 1^2 = 2\left(\frac{z}{c\sqrt{2}}\right)^2,$$

and, $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + (p - z)^2 = 2\left(\frac{p + z}{\sqrt{2}}\right)^2, \text{ respectively:}$$

also that rule and developable surfaces are excluded may be shewn thus. The equation $U + V + 2S = 0$ denotes a tangent plane, for eliminating S from (22) by means of this equation, we have

$$2T^2 + (U - V)^2 = 0,$$

which being the *sum* of two squares, requires that $T = 0$ and $U = V$; so that the plane, $U + V + 2S = 0$, meets the surface (22) only in the *point* denoted by $T = 0$ and $U = V = -S$, and it is therefore a tangent plane; but every tangent plane to a developable or rule surface meets the surface in one or two straight lines, hence (22) can denote umbilical surfaces only.*

Since $U + V + 2S = 0$ is the equation of a plane that passes through the edge (18, 20) of the hexahedron, it appears that $U + V + 2S = 0$ is the tangent plane passing

* I am indebted to Mr. Hearn for the remark that the equation (22) cannot denote rule surfaces. The simple proof given above is also due to this gentleman.

through this edge. Similarly, the equations to the tangent planes passing through the other edges may be obtained, and, tabulating the whole, we have

$$\left.\begin{array}{ll} U+V+2S=0, & U+V-2S=0 \\ U-V+2S=0, & U-V-2S=0 \\ V+T+2S=0, & V+T-2S=0 \\ V-T+2S=0, & V-T-2S=0 \\ T+U+2S=0, & T+U-2S=0 \\ T-U+2S=0, & T-U-2S=0 \end{array}\right\} \ldots\ldots(23).$$

It is easy from (16...21) to shew that the equations to the diagonal straight lines of the hexahedron are

$$\left.\begin{array}{r} T=\quad U=\quad V \\ -T=\quad U=\quad V \\ T=-U=\quad V \\ T=\quad U=-V \end{array}\right\} \ldots\ldots\ldots\ldots(24),$$

and that the equations to the diagonal planes are

$$\left.\begin{array}{l} U+V=0 \\ U-V=0 \\ V+T=0 \\ V-T=0 \\ T+U=0 \\ T-U=0 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots(25).$$

Again, the point in which the edge (18, 20) touches the surface (22) is evidently determined by the equations $T=0$ and $U=V=-S$; also the point of contact of the opposite edge (19, 21) is given by $T=0$ and $U=V=S$; hence the equations to the straight line joining these are $T=U-V=0$. In a similar manner we shall obtain the equations to the straight lines joining the points of contact of the other opposite edges, and collecting the whole, we have

$$\left.\begin{array}{l} T=U+V=0 \\ T=U-V=0 \\ U=V+T=0 \\ U=V-T=0 \\ V=T+U=0 \\ V=T-U=0 \end{array}\right\} \ldots\ldots\ldots\ldots(26).$$

From these equations it appears that each of the planes whose equations are $T = 0$, $U = 0$, and $V = 0$, passes through four points of contact, which are on edges that join opposite faces.

Now it is evident that the opposite faces (16, 17), (18, 19), and (20, 21) intersect in the plane $S = 0$; and that the diagonal straight lines (24), and the straight lines (26), joining the points of contact of opposite edges, pass through the point $T = U = V = 0$. Hence (I.), (II.), and (III.) are proved for the hexahedron.

It is evident from (23) that the opposite tangent planes passing through the edges of the hexahedron also intersect in straight lines in the plane $S = 0$, hence

IV. *If the edges of* $\left\{\begin{matrix}\textit{a hexahedron}\\ \textit{an octahedron}\end{matrix}\right\}$ *touch an umbilical surface of the second degree, and tangent planes be applied at the points of contact, the opposite faces and the opposite tangent planes intersect in straight lines in one plane.*

This, of course, has as yet been proved for the hexahedron only, but it will be shewn to be true for the octahedron presently.

Next, let the edges of an octahedron touch an umbilical surface of the second degree. Through the points of contact of the contiguous edges let planes be drawn; these planes (every opposite two excepted) will evidently intersect each other, two and two, in straight lines touching the surface, and they will therefore be the faces of a hexahedron whose edges touch the surface in the same points as the edges of the octahedron: also, since the hexahedron and octahedron are evidently so related that the faces of the one pass through the points of contact of the contiguous edges of the other, if (16...21) denote the faces of the hexahedron, we have only to find the equations to the planes that pass through the points of contact of every three contiguous edges of the hexahedron, and we shall have the equations to the faces of the octahedron.

Hence, (16...21) and (26), the equations to the faces of the octahedron are

$$T + U + V + 2S = 0 \ldots (27), \qquad T + U + V - 2S = 0 \ldots (28),$$
$$T - U - V + 2S = 0 \ldots (29), \qquad T - U - V - 2S = 0 \ldots (30),$$
$$-T + U - V + 2S = 0 \ldots (31), \qquad -T + U - V - 2S = 0 \ldots (32),$$
$$-T - U + V + 2S = 0 \ldots (33), \qquad -T - U + V - 2S = 0 \ldots (34),$$

the opposite faces being (27) and (28), (29) and (30), (31) and (32), and (33) and (34).

From these equations it may easily be shewn that the equations to the diagonal straight lines of the octahedron are

$$\left.\begin{array}{l} U = V = 0 \\ V = T = 0 \\ T = U = 0 \end{array}\right\} \ldots\ldots\ldots\ldots (35),$$

and hence the octahedron has three diagonal planes whose equations are

$$T = 0, \quad U = 0, \quad V = 0 \ldots\ldots\ldots\ldots (36).$$

The equation (22), of course, denotes the surface, which is therefore umbilical; also, since the points of contact of the edges of the octahedron coincide with those of the edges of the hexahedron described as above, the equations (26) denote the straight lines joining the points of contact of opposite edges of the octahedron, and the equations (23) denote the tangent planes at the said points.

Hence, (27...34), the opposite faces of the octahedron intersect in the plane $S = 0$, and, (23), the tangent planes passing through the opposite edges intersect in the same plane; also, (35) and (26), the diagonal straight lines, and the straight lines joining the points of contact of opposite edges, pass through the point $T = U = V = 0$. Consequently the theorems (I.), (II.), (III.), and (IV.) have now been completely established both for the hexahedron and the octahedron.

The following theorem, (V.), follows immediately from what has been said above, and the equations (16...21), (27...34), (23) and (24, 26, 35).

V. *If the edges of a hexahedron and of an octahedron touch an umbilical surface of the second degree in such a way that the faces of each pass through the points of contact on the contiguous edges of the other, and tangent planes be drawn at the points of contact; the opposite faces and the opposite tangent planes intersect in thirteen straight lines in one plane; also the seven diagonal straight lines and the six straight lines joining the points of contact of opposite edges pass through one point.**

Several interesting particulars respecting these solids might be given, but I am obliged to omit them.

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* I consider this theorem analogous to the following. If a quadrilateral be inscribed in a curve of the second degree, and another be circumscribed about the same, the points of contact of the sides of the latter coinciding with the angular points of the former, the four pairs of opposite sides will intersect in four points in one straight line, and the four diagonals will pass through one point. (*Nov.* 13, 1848).

The equations to the faces of the octahedron may also be written in other elegant forms. Assume $2T_1 = T + U + V$, $2U_1 = T - U - V$, $2V_1 = -T + U - V$, and $2W_1 = -T - U + V$, so that we have

$$T_1 + U_1 + V_1 + W_1 = 0 \text{ .... } (37)$$

identically, and the equations (27...34) become

$$\left.\begin{array}{ll} T_1 + S = 0, & T_1 - S = 0 \\ U_1 + S = 0, & U_1 - S = 0 \\ V_1 + S = 0, & V_1 - S = 0 \\ W_1 + S = 0, & W_1 - S = 0 \end{array}\right\} \text{ ...... } (38).$$

Also the equation (22) to the surface may be written

$$T_1^2 + U_1^2 + V_1^2 + W_1^2 = 2S^2 \text{ .... } (39).$$

The equations may be again transformed as follows: assume $S = t + u + v + w$, $T_1 = 3t - u - v - w$, $U_1 = 3u - t - v - w$, $V_1 = 3v - t - u - w$, and therefore, (37), $W_1 = 3w - t - u - v$; hence the equations to the faces of the octahedron may be put under the forms,

$$\left.\begin{array}{ll} t = 0, & -t + u + v + w = 0 \\ u = 0, & t - u + v + w = 0 \\ v = 0, & t + u - v + w = 0 \\ w = 0, & t + u + v - w = 0 \end{array}\right\} \text{ ...... } (40),*$$

and the equation to the surface is

$$t^2 + u^2 + v^2 + w^2 = \tfrac{3}{8}(t + u + v + w)^2 \text{ ..... } (41).$$

We now come to the converse theorems.

VI. *If the opposite faces of* $\left\{\begin{array}{l}\textit{a hexahedron} \\ \textit{an octahedron}\end{array}\right\}$ *intersect in* $\left\{\begin{array}{l}\textit{three} \\ \textit{four}\end{array}\right\}$ *straight lines in one plane, the edges of the solid figure will touch an umbilical surface of the second degree.*

VII. *If the* $\left\{\begin{array}{l}\textit{four} \\ \textit{three}\end{array}\right\}$ *straight lines joining the opposite*

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* The equations to the faces of the hexahedron might also be written,

$$\begin{array}{ll} t + u = 0, & v + w = 0, \\ t + v = 0, & u + w = 0, \\ t + w = 0, & u + v = 0, \end{array}$$

and then $\quad t^2 + u^2 + v^2 + w^2 = \tfrac{3}{4}(t + u + v + w)^2$

would be the equation to the surface touching its edges.

angular points of $\left\{\begin{matrix}\textit{a hexahedron}\\ \textit{an octahedron}\end{matrix}\right\}$ *intersect in a point, the edges of the solid figure will touch an umbilical surface of the second degree.*

Let (1, 2, 3) denote three contiguous faces of a hexahedron whose opposite faces intersect in the plane $S = 0$; then, supposing t, u, and v to have been multiplied by the proper constants, we may denote the faces opposite to (1), (2), and (3) respectively by

$$t - 2S = 0, \quad u - 2S = 0, \quad v - 2S = 0,$$

and if in these equations and (1, 2, 3) we write $T + S$, $U + S$, and $V + S$, for t, u, and v, we shall obtain the equations (16....21), and hence the edges of the hexahedron touch the surface (22) which is umbilical, and of the second degree.

Again, let (1, 2, 3, 5, 6, 7) denote the faces of a hexahedron, whose diagonal straight lines intersect in a point. The opposite edges (1, 2) and (5, 6) are evidently in the same plane, but this can only be when s and v can be eliminated from (5) and (6) at the same time, and this requires $c_1 = c_2$; similarly, $b_1 = b_3$, and $a_2 = a_3$; hence, if we put $2S = a_2t + b_1u + c_1v - s$, and then write t', u', and v', for $(a_2 - 1)t$, $(b_1 - 1)u$, and $(c_1 - 1)v$, the equations (1, 2, 3, 5, 6, 7) will become

$$t' = 0, \quad u' = 0, \quad v' = 0,$$

$$t' - 2S = 0, \quad u' - 2S = 0, \quad v' - 2S = 0;$$

from which we get the same result as above; and the theorems (VI.) and (VII.) are established for the hexahedron.

Next, let the faces of an octahedron be denoted by

$$t = 0....(42), \qquad -t + u + v + w = 0....(43),$$

$$u = 0....(44), \qquad t + au + bv + cw = 0....(45),$$

$$v = 0....(46), \qquad t + a_1u + b_1v + c_1w = 0....(47),$$

$$w = 0....(48), \qquad t + a_2u + b_2v + c_2w = 0....(49),$$

the opposite faces being (42) and (43), &c.

Since four faces of an octahedron intersect in every angular point, each of the following sets of equations must denote a point; namely, (43, 45, 46, 48), (43, 44, 47, 48), (43, 44, 46, 49), (42, 44, 47, 49), (42, 45, 46, 49), and (42, 45, 47, 48); hence the following conditions, $a = b_1 = c_2 = -1$, and $b_2c_1 = a_2c = a_1b = 1$; assume $c_1 = \lambda$, $a_2 = \mu$, and

$b = \nu$, hence, $b_2 = \lambda^{-1}$, $c = \mu^{-1}$, and $a_1 = \nu^{-1}$. It is therefore evident that the faces of *any* octahedron may be denoted by the equations,

$$\left.\begin{array}{ll} t = 0, & -t + u + v + w = 0 \\ u = 0, & t - u + \nu v + \mu^{-1} w = 0 \\ v = 0, & t + \nu^{-1} u - v + \lambda w = 0 \\ w = 0, & t + \mu u + \lambda^{-1} v - w = 0 \end{array}\right\} \ldots\ldots(50);$$

and from these it is easy to shew that the equations to the diagonal straight lines are,

$$\left.\begin{array}{l} t - u = v - \lambda w = 0 \\ t - v = w - \mu u = 0 \\ t - w = u - \nu v = 0 \end{array}\right\} \ldots\ldots\ldots\ldots (51).$$

Now it is evident that if the opposite faces intersect in straight lines in a plane, we must, (50), have $\lambda = \mu = \nu = 1$ (for we evidently cannot have $\lambda = \mu = \nu = -1$); or if the diagonals pass through a point, we must, (51), have $\lambda = \mu = \nu = 1$. In each case the equations (50) coincide with (40), and hence the edges of the octahedron touch an umbilical surface of the second degree whose equation is (41).

Before considering those theorems which for the rule and developable surfaces of the second degree are analogous to those of Pascal and Brianchon, I shall notice two or three properties intimately connected with what has been already given, but these I propose to dispose of as briefly as possible.

The equation to any surface of the second degree touching the faces (16 .... 21) of a hexahedron whose diagonals intersect in a point, is

$$\sin^2\theta\, T^2 + \sin^2\phi\, U^2 + \sin^2\psi\, V^2 + 2\cos\theta \sin\phi \sin\psi\, UV,$$
$$+ 2\cos\phi \sin\psi \sin\theta\, TV + 2\cos\psi \sin\theta \sin\phi\, TU,$$
$$= (1 - \cos^2\theta - \cos^2\phi - \cos^2\psi + 2\cos\theta\cos\phi\cos\psi)\, S^2 \ldots (52),$$

θ, ϕ, and ψ being arbitrary constants.

Put $h = \dfrac{\sin\phi \sin\psi}{\cos\theta - \cos\phi\cos\psi}$, $k = \dfrac{\sin\theta\sin\psi}{\cos\phi - \cos\theta\cos\psi}$,

and $$l = \frac{\sin\theta\sin\phi}{\cos\psi - \cos\theta\cos\phi},$$

then it may readily be shewn that the equations to the three lines joining the points of contact of opposite faces are,

$$\left.\begin{array}{l} -T = lU = kV \\ -U = lT = hV \\ -V = kT = hU \end{array}\right\} \ldots\ldots\ldots\ldots (53);$$

and these lines, whatever may be the values of θ, ϕ, and ψ, always pass through the point $T = U = V = 0$, which is the point in which the diagonals intersect.

Again, the equation to any surface of the second degree touching the faces (27 .... 34) of an octahedron whose diagonals intersect in a point, is

$$\left.\begin{array}{l} mT^2 + nU^2 + pV^2 = 4S^2 \\ m^{-1} + n^{-1} + p^{-1} = 1 \end{array}\right\} \ldots\ldots\ldots (54);$$

where

and the equations to the lines joining the points of contact of opposite faces are easily shewn to be,

$$\left.\begin{array}{l} mT = nU = pV \\ -mT = nU = pV \\ mT = -nU = pV \\ mT = nU = -pV \end{array}\right\} \ldots\ldots\ldots\ldots (55),$$

and these equations being satisfied by $T = U = V = 0$, shew that the straight lines (55) and the diagonals always intersect in one point.

We may also present the equation to any surface of the second degree touching the faces of the octahedron in a form different from (54); thus, taking (40) to denote the said faces, the equation to the surface will take the form,

$$\left.\begin{array}{l} t^2 + u^2 + v^2 + w^2 + 2\cos\alpha(tu + vw) + 2\cos\beta(tv + uw) + 2\cos\gamma(tw + uv) = 0 \\ \text{where} \quad \alpha + \beta + \gamma = 2\pi \end{array}\right\}$$

$$\ldots\ldots (56).^*$$

From what has been already said, the following theorem is evident,

VIII. *If the diagonal straight lines of* $\begin{Bmatrix} \textit{a hexahedron} \\ \textit{an octahedron} \end{Bmatrix}$ *intersect in a point, and any surface of the second degree be*

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* The equations (52), (54), and (56) may easily be verified. It would occupy more space than is consistent with the plan of this paper, to give at present the analysis by which they were obtained. I may observe, however, that a part at least of the investigations is very similar to that employed by me in the *Mathematician* (vol. II. pp. 261, 262) to obtain the equations to surfaces of the second degree inscribed in a tetrahedron or parallelepiped, these equations being indeed identical in form with (52) and (56).

described to touch the faces, the straight lines joining the points of contact of opposite faces pass through the intersection of the diagonal straight lines.

This may also be deemed an analogue of the plane theorem to which I consider (III.) analogous, and indeed we may include (III.) and (VIII.) in one enunciation, as follows:

IX. *If any surface of the second degree be described to touch the faces of* $\left\{\begin{matrix}\textit{a hexahedron}\\ \textit{an octahedron}\end{matrix}\right\}$ *whose edges touch an umbilical surface of the second degree, then shall the straight lines joining the opposite angular points of the solid figure, those joining the points of contact of opposite edges, and those joining the points of contact of opposite faces, pass through one point.*

Conversely,

X. *If the faces of* $\left\{\begin{matrix}\textit{a hexahedron}\\ \textit{an octahedron}\end{matrix}\right\}$ *touch a surface of the second degree in such a way that the straight lines joining the points of contact of opposite faces intersect in a point, the edges will touch an umbilical surface of the second degree.**†

This follows at once from (VI.) and the following theorem, which is an easy consequence of known properties of surfaces of the second degree. If the straight line joining the points of contact of a pair of tangent planes to a surface of the second degree always passes through a fixed point, the tangent planes will always intersect in a fixed plane.

Recurring again to the investigation by which the equation (22) was obtained, it will be seen that if none of the quantities λ, μ, and ν be zero, the equation to surfaces touching the edges of hexahedra or octahedra will necessarily have the form (22), and the surfaces will therefore be umbilical. But

* The following passage occurs in a letter (dated April 6th, 1848) which I received from Mr. Hearn a short time before I discovered the theorems given in this paper.

"If *four* points be assumed on a surface of the second order (not in the same plane), two other points on the surface may be found such that the six points shall be the angular points of an inscribed octahedron, whose opposite faces meet in four straight lines, which intersect each other two and two, and are in one plane; and also such that if tangent planes to the surface are drawn at the said six points, they will form a circumscribing hexahedron, the lines joining the opposite angles of which shall pass through the same point."

† The hexahedron and octahedron having diagonal planes, possess properties analogous to certain theorems connected with the complete quadrilateral (see Salmon's *Conic Sections*, Art. 238); to which I shall take some other opportunity to refer more particularly. (*Nov.* 13, 1848.)

if some of the quantities λ, μ, ν be zero, so that the surface is developable, such solid figures may be described, but they will not always possess the properties (I.) (II.) (III.).* Hence (in the words of Mr. Hearn) "If a hexahedron or octahedron be described, whose edges touch a surface of the second order, it will necessarily have diagonal planes, provided the surface is not a cone or cylinder. The surfaces for which the construction is possible are the ellipsoid, elliptic paraboloid and hyperboloid of two sheets. . . . Those for which the construction is impossible are the hyperboloid of one sheet, and the hyperbolic paraboloid. About cones and cylinders such solids may be described, but they have not necessarily diagonal planes."

We are still, therefore, in want of theorems for the rule surfaces analogous to those of Pascal and Brianchon, and these I conceive to be the following:

Let six planes (which of course will be tangent planes) intersect two and two in order in (six) straight lines in a rule surface of the second degree, then,

XI. *The opposite planes intersect in three straight lines in one plane.*

XII. *The three straight lines joining the points of contact† of opposite planes intersect in a point, and the system has three diagonal planes.*

In the rule surfaces of the second degree we know that no two rectilinear generators that belong to the same system are in one plane, while any two generators of different systems are in the same plane. Now in the present case the six lines of intersection are evidently rectilinear generators, and any adjacent two of them, being in one plane, must belong to different systems; hence the alternate intersections belong to the same system, and any opposite two will therefore belong to different systems, and be in one plane. This proves that the given system of planes has three diagonal planes.

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* Thus, if the equations to the faces of a hexahedron be

$$\begin{array}{lllll} t = 0, & & t + (1+\rho)\,u + \mu v = s, \\ u = 0, & & (1-\rho)\,t + u + \lambda v = s, \\ v = 0, & & \mu t + \lambda u + v = s, \end{array}$$

its edges will touch the developable surface, whose equation is

$$s^2 = 4\,(\mu t + \lambda u)\,v\,;$$

and yet this hexahedron does not possess the properties referred to above.

† The points of contact are also the points in which the six given planes intersect three and three in order.

Supposing t, u, and v to have been multiplied by constants, we may denote the six planes by

$$t = 0 \ldots (57), \qquad (a-1)t + bu + cv + 1 = 0 \ldots (60),$$

$$u = 0 \ldots (59), \qquad at + (b-1)u + c_1 v + 1 = 0 \ldots (62),$$

$$v = 0 \ldots (61), \qquad a_1 t + b_1 u + (c-1)v + 1 = 0 \ldots (58),$$

these equations being numbered in the order of the planes, and those of the opposite planes being placed in the same horizontal line.

Now it has just been shewn that the lines (57, 58) and (60, 61) are in one plane, hence we must have $b_1 = b$; similarly considering the other opposite lines of intersection, we have $a_1 = a$, and $c_1 = c$. Hence if we put $2S = at + bu + cv + 1$, and then substitute $T + S$ for t, $U + S$ for u, and $V + S$ for v, the equations to the planes will become,

$$T + S = 0 \ldots\ldots (63), \qquad T - S = 0 \ldots\ldots (66),$$

$$U + S = 0 \ldots\ldots (65), \qquad U - S = 0 \ldots\ldots (68),$$

$$V + S = 0 \ldots\ldots (67), \qquad V - S = 0 \ldots\ldots (64).$$

Hence the equations to the diagonal planes which pass through the lines (63, 64) and (66, 67), (64, 65) and (67, 68), and (65, 66) and (68, 63), respectively, are

$$\left.\begin{aligned} T + V &= 0 \\ U + V &= 0 \\ U + T &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (69).$$

Now recollecting that the lines (63, 64) and (66, 67) are in the plane $T + V = 0$, it is evident that the equation to the rule surface of the second degree, passing through the lines (63, 64), (64, 65), (65, 66), and (66, 67), is

$$(T - S)(V - S) + \lambda (U + S)(T + V) = 0,$$

and the equation to the rule surface that passes through the lines (66, 67), (67, 68), (68, 63) and (63, 64) is,

$$(T + S)(V + S) + \mu (U - S)(T + V) = 0;$$

and that these surfaces may coincide, it is necessary and sufficient that $\lambda = \mu = 1$, and then either of the preceding equations becomes

$$S^2 + TU + TV + UV = 0 \ldots\ldots\ldots (70),$$

which therefore is the equation to the given rule surface.

Moreover, eliminating T from (70) by means of (63), we have $(U - S)(V - S) = 0$; hence the point in which (63) touches the surface is determined by the equations $T + S = 0$, $U - S = 0$, and $V - S = 0$, or by $-T = U = V = S$; similarly the equations to the point of contact (66) and (70) are $-T = U = V = -S$; consequently the line joining these points is denoted by $-T = U = V$. Hence the equations to the three straight lines that join the points of contact of opposite faces are,

$$\left.\begin{array}{r} -T = U = V \\ T = -U = V \\ T = U = -V \end{array}\right\} \ldots\ldots\ldots\ldots (71).$$

Now the opposite planes (63, 66), (64, 67), and (65, 68) intersect in the plane $S = 0$; also the straight lines (71) and the diagonal planes (69) all pass through the point $T = U = V = 0$; consequently (XI.) and (XII.) are fully established.

We might have proceeded rather differently. Let the given surface and the six planes be cut by any two planes (parallel ones for example), each of these two planes will intersect the system in a hexagon inscribed in a conic section, and, by Pascal's theorem, the opposite sides of each hexagon will intersect in three points in a straight line; hence the given planes, which pass through the sides of the hexagons, intersect in three straight lines in one plane. It might now easily be shewn, as was virtually done in establishing (VI.), that the planes may be denoted by the equations (63. . . . 68), and the remainder of the investigation will be the same as above.

It is easy to construct a system of six planes which shall intersect two and two in order in a given rule surface of the second degree. We have only to take three straight lines, G_1, G_3, G_5, in the surface belonging to the same system of rectilinear generators, and three, g_2, g_4, g_6, belonging to the other system of generators; then shall (G_1g_2), (g_2G_3), (G_3g_4), (g_4G_5), (G_5g_6), and (g_6G_1) be such a system of planes. (By the plane (Gg) I mean the plane passing through the intersecting lines G and g).

The converses of (XI.) and (XII.) are as follows:—

XIII. *If six planes intersect two and two in order in six straight lines so that no two of the alternate lines of intersection may be in one plane, and so that the opposite planes intersect in three straight lines in one plane, then a rule*

surface of the second degree may be made to pass through the six lines of intersection.

XIV. *If six planes intersect three and three in order in six points so that the three straight lines joining opposite points intersect in a point, a rule surface of the second degree may be made to pass through the six lines in which the planes intersect two and two in order.*

These two theorems are obvious from what has been already said; it is only necessary to remember that in (XIV.) every two opposite lines of intersection are evidently in one plane, and the system has three diagonal planes.

It is not a little curious, that though the analogues of Pascal's and Brianchon's theorems for the rule surfaces of the second degree are so different from those for the umbilical surfaces, yet (regarding the hexahedron only) we are led in both cases to the same system of *planes*. This remark suggests several interesting relations, but it will be sufficient to notice one or two. If the edges of a hexahedron touch an umbilical surface of the second degree, then taking the faces of the hexahedron in any order, so that however the opposite faces shall be opposite still, a rule surface of the second degree may be drawn through the six lines in which the successive faces intersect; and conversely. Again, if the opposite faces of a hexahedron intersect in three straight lines in one plane, an umbilical surface of the second degree may be described to touch its edges; and if the faces be taken in order as before, a rule surface of the second degree may be drawn through the six lines in which the successive faces intersect.

We come, finally, to the analogues for the developable surfaces of the second degree. These are obvious enough from the plane theorems themselves, and they are, I believe, well known; but I am not aware it has been noticed before that the enunciations are capable of being so modified that the properties shall be true for *all* the surfaces of the second degree.

XV. *If the lateral edges of a hexagonal prism or pyramid lie in a developable surface, or touch a non-developable surface of the second degree, the opposite lateral faces will intersect in three straight lines in one plane.*

XVI. *If the lateral faces of a hexagonal prism or pyramid touch any surface of the second degree, the three planes passing through the opposite lateral edges will intersect in a straight line.*

The truth of these two theorems is easily established. Suppose the surface and solid figure to be cut by the plane which passes through the six points of contact (or by *any* plane, if the surface be developable), then in the case of (XV.) we shall have a hexagon inscribed in a curve of the second degree, the opposite sides of which will, by Pascal's theorem, intersect in three points in a straight line; hence the opposite lateral faces of the prism or pyramid will intersect in three straight lines in one plane. Also in the case of (XVI.) we shall have a hexagon whose sides touch a curve of the second degree, and whose diagonals joining opposite angular points will therefore intersect in a point; hence the three planes passing through the opposite lateral edges of the prism or pyramid will intersect in a straight line.

Conversely,

XVII. *If the opposite lateral faces of a hexagonal prism or pyramid intersect in three straight lines in one plane, the lateral edges will lie in a developable surface of the second degree, and will touch an infinite number of non-developable surfaces of the second degree.*

XVIII. *If the three planes passing through the opposite lateral edges of a hexagonal prism or pyramid intersect in a straight line, the lateral faces will touch a developable surface, and an infinite number of non-developable surfaces of the second degree.*

For, in (XVII.), any plane will intersect the prism or pyramid in a hexagon whose opposite sides intersect in three points in a straight line; hence, by the converse of Pascal's theorem, a curve of the second degree may be circumscribed about the hexagon, and consequently the edges of the solid figure lie in a developable surface of the second degree; let $P = 0$ be the equation to this developable surface, and $q = 0$ be the equation to *any* plane, then (Gregory's *Solid Geometry*, p. 123) the non-developable surface, whose equation is $P + \lambda q^2 = 0$, will touch the developable surface along the curve in which the latter surface is intersected by the plane $q = 0$, and this non-developable surface will therefore touch the edges of the solid figure.

In like manner we may, by aid of the converse of Brianchon's theorem, shew that, in (XVIII.), the faces of the solid figure touch a developable surface of the second degree, and that if $P = 0$ be the equation to this surface, and $q = 0$ be the equation to any plane, the non-developable surface, whose

equation is $P + \lambda q^2 = 0$, will also touch the faces of the solid figure.

Wimbledon, Surrey, June 7, 1848.

NOTE. Since the preceding paper was written, I have been informed by Mr. Cayley that the theorems given above for the rule and developable surfaces are to be found, in substance, in a memoir of Hesse's, entitled "Ueber das geradlinige Rechteck auf dem Hyperboloid." (*Crelle*, t. XXIV.)

I find too that M. Chasles, in his "Aperçu Historique," has given certain theorems relative to the tetrahedron, which may be viewed as analogous to the theorems of Pascal and Brianchon. I recently rediscovered some of Chasles's theorems, but having been informed by a friend that he believed they had already appeared in the work referred to, I ventured to apply to that distinguished geometer, who most kindly and promptly furnished me with a copy of his theorems, and from these I find that I have been anticipated to a considerable extent, though not entirely so. Possibly I may have something to say on this subject hereafter.

November 13, 1848.

ABSTRACT OF A MEMOIR BY DR. HESSE ON THE CONSTRUCTION OF THE SURFACE OF THE SECOND ORDER WHICH PASSES THROUGH NINE GIVEN POINTS.

By ARTHUR CAYLEY.

THE construction to be presently given of the surface of the second order which passes through nine given points, is taken from a memoir by Dr. Hesse (*Crelle*, tom. XXIV. p. 36). It depends upon the following lemma, which is there demonstrated.

LEMMA. The polar plane of a fixed point P with respect to any surface of the second order passing through seven given points, passes through a fixed point Q (which may be termed the harmonic pole of the point P with respect to the system of surfaces of the second order).

PROBLEM. Given the seven points 1, 2, 3, 4, 5, 6, 7, and a point P, to construct the harmonic pole Q of the point P with respect to the system of surfaces of the second order passing through the seven points.

The required point Q may be considered as the intersection of the polar planes of the point P with respect to any three hyperboloids, each of which passes through the seven given points; any such hyperboloid may be considered as deter-

mined by means of three of its generating lines. These considerations lead to the construction following.

1. Connecting the points 1 and 2, and also the points 3 and 4, by two straight lines, and determining the three lines, each of which passes through one of the points 5, 6, 7, and intersects both of the first-mentioned lines, the three lines so determined are generating lines of a hyperboloid passing through the seven points.

Two other systems of generating lines (belonging to two new hyperboloids) are determined by the like construction, interchanging the points 1, 2, 3, 4. And by interchanging all the seven points we obtain 105 systems of generating lines (belonging to as many different hyperboloids, unless some of these hyperboloids are identical).

2. It remains to be shewn how the polar plane of the point P with respect to one of the 105 hyperboloids may be constructed. Drawing through the point P three lines, each of which passes through two of the three given generating lines of the hyperboloid in question, the points of intersection of the lines so determined with the generating lines which they respectively intersect, are points of the hyperboloid. Hence, constructing upon each of the three lines in question the harmonic pole of the point P with respect to the two points of intersection, the plane passing through the three harmonic poles is the polar plane of P with respect to the hyperboloid. Hence, constructing the polar planes of P with respect to any three of the 105 hyperboloids, the point of intersection of these three polar planes is the required point Q.

Problem. To construct the polar plane of a point P with respect to the surface of the second order which passes through nine given points 1, 2, 3, 4, 5, 6, 7, 8, 9.

Consider any seven of the nine points, *e.g.* the points 1, 2, 3, 4, 5, 6, 7, and construct the harmonic pole of the point P with respect to the system of surfaces of the second order passing through these seven points. By permuting the different points we obtain 36 different points Q, all of which lie in the same plane. This plane (which is of course determined by any three of the thirty-six points) is the required polar plane. Hence we obtain the solution of

Problem. To construct the surface of the second order which passes through nine given points 1, 2, 3, 4, 5, 6, 7, 8, 9.

Assuming the point P arbitrarily, construct the polar plane of this point with respect to the surface of the second order passing through the nine points. Join the point P with any one of the nine points, *e.g.* the point 1, and on the line so formed determine the harmonic pole R of the point 1 with respect to the point P, and the point where the line $P1$ is intersected by the polar plane. R is a point of the required surface of the second order, which surface is therefore determined by giving every possible position to the point P.

This construction is the complete analogue of Pascal's theorem *considered as a construction for describing the conic section which passes through five given points*. And it would appear that the principles by means of which the construction is obtained ought to lead to the analogue of Pascal's theorem considered in its ordinary form, *i.e. as a relation between six points of a conic*, or in other words to the solution of the problem to determine the relation between ten points of a surface of the second order; but this problem, one of the most interesting in the theory of surfaces of the second order, remains as yet unsolved. The problem last mentioned was proposed as a prize question by the Brussels Academy, who subsequently proposed the more general question to determine the analogue for surfaces of the second order of Pascal's theorem. This of course admitted of being answered in a variety of different ways, according to the different ways of viewing the theorem of Pascal. Thus, M. Chasles, considering Pascal's theorem as a property of a conic intersected by the three sides of a triangle, discovered the following very elegant analogous theorem for surfaces of the second order.

"The six edges of a tetrahedron may be considered as intersecting a surface of the second order in twelve points lying three and three upon four planes, each one of which contains three points lying on edges which pass through the same angle of the tetrahedron; these planes meet the faces opposite to these angles in four straight lines which are generating lines (of the same species) of a certain hyperboloid."

It is hardly necessary to remark that all the properties involved in the present memoir are such as to admit of being transformed by the theory of reciprocal polars.

ON THE SIMULTANEOUS TRANSFORMATION OF TWO HOMOGENEOUS FUNCTIONS OF THE SECOND ORDER.

By Arthur Cayley.

The theory of the simultaneous transformation by linear substitutions of two homogeneous functions of the second order has been developed by Jacobi in the memoir "De binis quibuslibet functionibus," &c. *Crelle*, tom. XII. p. 1; but the simplest method of treating the problem is the one derived from Mr. Boole's Theory of Linear Transformations, combined with the remark in his "Notes on Linear Transformations," in this *Journal*, vol. IV. p. 167, (Old Series). As I shall have occasion to refer to the results of this theory in the second part of my paper "On the Attraction of Ellipsoids", in the present number of the Journal, I take this opportunity of developing the formula in question; considering for greater convenience the case of three variables only.

Suppose that by a linear transformation,

$$x = \alpha x_1 + \beta y_1 + \gamma z_1,$$

$$y = \alpha' x_1 + \beta' y_1 + \gamma' z_1,$$

$$z = \alpha'' x_1 + \beta'' y_1 + \gamma'' z_1,$$

we have identically,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy$$

$$= a_1 x_1^2 + b_1 y_1^2 + c_1 z_1^2 + 2f_1 y_1 z_1 + 2g_1 z_1 x_1 + 2h_1 x_1 y_1,$$

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy$$

$$= A_1 x_1^2 + B_1 y_1^2 + C_1 z_1^2 + 2F_1 y_1 z_1 + 2G_1 z_1 x_1 + 2H_1 x_1 y_1.$$

Of course, whatever be the values of a, b, c, f, g, h, the same transformation gives

$$\mathrm{a}x^2 + \mathrm{b}y^2 + \mathrm{c}z^2 + 2\mathrm{f}yz + 2\mathrm{g}zx + 2\mathrm{h}xy$$

$$= \mathrm{a}_1 x_1^2 + \mathrm{b}_1 y_1^2 + \mathrm{c}_1 z_1^2 + 2\mathrm{f}_1 y_1 z_1 + 2\mathrm{g}_1 z_1 x_1 + 2\mathrm{h}_1 x_1 y_1.$$

Provided that we have

$$\left\{\begin{aligned}
\mathrm{a}_1 &= \mathrm{a}\alpha^2 + \mathrm{b}\alpha'^2 + \mathrm{c}\alpha''^2 + 2\mathrm{f}\alpha'\alpha'' + 2\mathrm{g}\alpha''\alpha + 2\mathrm{h}\alpha\alpha', \\
\mathrm{b}_1 &= \mathrm{a}\beta^2 + \mathrm{b}\beta'^2 + \mathrm{c}\beta''^2 + 2\mathrm{f}\beta'\beta'' + 2\mathrm{g}\beta''\beta + 2\mathrm{h}\beta\beta', \\
\mathrm{c}_1 &= \mathrm{a}\gamma^2 + \mathrm{b}\gamma'^2 + \mathrm{c}\gamma''^2 + 2\mathrm{f}\gamma'\gamma'' + 2\mathrm{g}\gamma''\gamma + 2\mathrm{h}\gamma\gamma', \\
\mathrm{f}_1 &= \mathrm{a}\beta\gamma + \mathrm{b}\beta'\gamma' + \mathrm{c}\beta''\gamma'' + \mathrm{f}(\beta'\gamma'' + \beta''\gamma') + \mathrm{g}(\beta''\gamma + \beta\gamma'') + \mathrm{h}(\beta\gamma' + \beta'\gamma), \\
\mathrm{g}_1 &= \mathrm{a}\gamma\alpha + \mathrm{b}\gamma'\alpha' + \mathrm{c}\gamma''\alpha'' + \mathrm{f}(\gamma'\alpha'' + \gamma''\alpha') + \mathrm{g}(\gamma''\alpha + \gamma\alpha'') + \mathrm{h}(\gamma\alpha' + \gamma'\alpha), \\
\mathrm{h}_1 &= \mathrm{a}\alpha\beta + \mathrm{b}\alpha'\beta' + \mathrm{c}\alpha''\beta'' + \mathrm{f}(\alpha'\beta'' + \alpha''\beta') + \mathrm{g}(\alpha''\beta + \alpha\beta'') + \mathrm{h}(\alpha\beta' + \alpha'\beta).
\end{aligned}\right.$$

Representing for a moment the equations between the pairs of functions of the second order by

$$u = u_1, \quad U = U_1, \quad v = v_1,$$

we have, whatever be the value of λ,

$$\lambda u + U + v = \lambda u_1 + U_1 + v_1.$$

Whence, if

$$\begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix} = \Pi;$$

$$\begin{vmatrix} \lambda a_1 + A_1 + \mathrm{a}_1, & \lambda h_1 + H_1 + \mathrm{h}_1, & \lambda g_1 + G_1 + \mathrm{g}_1 \\ \lambda h_1 + H_1 + \mathrm{h}_1, & \lambda b_1 + B_1 + \mathrm{b}_1, & \lambda f_1 + F_1 + \mathrm{f}_1 \\ \lambda g_1 + G_1 + \mathrm{g}_1, & \lambda f_1 + F_1 + \mathrm{f}_1, & \lambda c_1 + C_1 + \mathrm{c}_1 \end{vmatrix}$$

$$= \Pi^2 \begin{vmatrix} \lambda a + A + \mathrm{a}, & \lambda h + H + \mathrm{h}, & \lambda g + G + \mathrm{g} \\ \lambda h + H + \mathrm{h}, & \lambda b + B + \mathrm{b}, & \lambda f + F + \mathrm{f} \\ \lambda g + G + \mathrm{g}, & \lambda f + F + \mathrm{f}, & \lambda c + C + \mathrm{c} \end{vmatrix}$$

Whence, since a, b, c, f, g, h, are arbitrary,

$$\begin{vmatrix} \lambda a_1 + A_1, & \lambda h_1 + H_1, & \lambda g_1 + G_1 \\ \lambda h_1 + H_1, & \lambda b_1 + B_1, & \lambda f_1 + F_1 \\ \lambda g_1 + G_1, & \lambda f_1 + F_1, & \lambda c_1 + C_1 \end{vmatrix} = \Pi^2 \begin{vmatrix} \lambda a + A, & \lambda h + H, & \lambda g + G \\ \lambda h + H, & \lambda b + B, & \lambda f + F \\ \lambda g + G, & \lambda f + F, & \lambda c + C \end{vmatrix}$$

which determine the relations which must subsist between the coefficients of the functions of the second order. We derive

$$\begin{vmatrix} a_1, & h_1, & g_1 \\ h_1, & b_1, & f_1 \\ g_1, & f_1, & c_1 \end{vmatrix} = \Pi^2 \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$$

and by comparing the coefficients of a, &c , if we write for shortness,

$$\mathfrak{A} = \begin{vmatrix} 1 & . & . \\ . & \lambda b + B, & \lambda f + F \\ . & \lambda f + F, & \lambda c + C \end{vmatrix} \text{ \&c.,}$$

we find

$$\mathfrak{A}_1 \alpha^2 + \mathfrak{B}_1 \beta^2 + \mathfrak{C}_1 \gamma^2 + 2\mathfrak{F}_1 \beta\gamma + 2\mathfrak{G}_1 \gamma\alpha + 2\mathfrak{H}_1 \alpha\beta = \Pi^2 \mathfrak{A},$$

$$\mathfrak{A}_1 \alpha'^2 + \mathfrak{B}_1 \beta'^2 + \mathfrak{C}_1 \gamma'^2 + 2\mathfrak{F}_1 \beta'\gamma' + 2\mathfrak{G}_1 \gamma'\alpha' + 2\mathfrak{H}_1 \alpha'\beta' = \Pi^2 \mathfrak{B},$$

$$\mathfrak{A}_1 \alpha''^2 + \mathfrak{B}_1 \beta''^2 + \mathfrak{C}_1 \gamma''^2 + 2\mathfrak{F}_1 \beta''\gamma'' + 2\mathfrak{G}_1 \gamma''\alpha'' + 2\mathfrak{H}_1 \alpha''\beta'' = \Pi^2 \mathfrak{C},$$

$$\mathfrak{A}_1\alpha'\alpha'' + \mathfrak{B}_1\beta'\beta'' + \mathfrak{C}_1\gamma'\gamma'' + \mathfrak{F}_1(\beta'\gamma'' + \beta''\gamma') + \mathfrak{G}_1(\gamma'\alpha'' + \gamma''\alpha') + \mathfrak{H}_1(\alpha'\beta'' + \alpha''\beta') = \Pi^2\mathfrak{F},$$

$$\mathfrak{A}_1\alpha''\alpha + \mathfrak{B}_1\beta''\beta + \mathfrak{C}_1\gamma''\gamma + \mathfrak{F}_1(\beta''\gamma + \beta\gamma'') + \mathfrak{G}_1(\gamma''\alpha + \gamma\alpha'') + \mathfrak{H}_1(\alpha''\beta + \alpha\beta'') = \Pi^2\mathfrak{G},$$

$$\mathfrak{A}_1\alpha\alpha' + \mathfrak{B}_1\beta\beta' + \mathfrak{C}_1\gamma\gamma' + \mathfrak{F}_1(\beta\gamma' + \beta'\gamma) + \mathfrak{G}_1(\gamma\alpha' + \gamma'\alpha) + \mathfrak{H}_1(\alpha\beta' + \alpha'\beta) = \Pi^2\mathfrak{H},$$

each of which virtually contains three equations on account of the indeterminate quantity λ. A somewhat more elegant form may be given to these equations; thus the first of them is

$$\begin{vmatrix} & \alpha, & \beta, & \gamma, \\ \alpha, & \lambda a_1 + A_1, & \lambda h_1 + H_1, & \lambda g_1 + G_1 \\ \beta, & \lambda h_1 + H_1, & \lambda b_1 + B_1, & \lambda f_1 + F_1 \\ \gamma, & \lambda g_1 + G_1, & \lambda f_1 + F_1, & \lambda c_1 + C_1 \end{vmatrix} = \Pi^2 \begin{vmatrix} 1 & . & . \\ . & \lambda b + B, & \lambda f + F \\ . & \lambda f + F, & \lambda c + C \end{vmatrix}$$

from which the form of the whole system is sufficiently obvious. The actual values of the coefficients α, β, &c. can only be obtained in the particular case, where $f_1 = g_1 = h_1 = F_1 = G_1 = H_1 = 0$. If we suppose besides (which is no additional loss of generality) that $a_1 = b_1 = c_1 = 1$, then the whole system of formulæ becomes

$$(A_1 + \lambda)(B_1 + \lambda)(C_1 + \lambda) = \Pi^2 \begin{vmatrix} \lambda a + A, & \lambda h + H, & \lambda g + G \\ \lambda h + H, & \lambda b + B, & \lambda f + F \\ \lambda g + G, & \lambda f + F, & \lambda c + C \end{vmatrix}$$

$$1 = \Pi^2 \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} \text{ or } \Pi^2 = \frac{1}{\kappa} \text{ suppose,}$$

$$(B_1+\lambda)(C_1+\lambda)\,\alpha^2 + (C_1+\lambda)(A_1+\lambda)\,\beta^2 + (A_1+\lambda)(B_1+\lambda)\,\gamma^2 = \frac{1}{\kappa}\,\mathfrak{A}$$

$$(B_1+\lambda)(C_1+\lambda)\,\alpha'^2 + (C_1+\lambda)(A_1+\lambda)\,\beta'^2 + (A_1+\lambda)(B_1+\lambda)\,\gamma'^2 = \frac{1}{\kappa}\,\mathfrak{B}$$

$$(B_1+\lambda)(C_1+\lambda)\,\alpha''^2 + (C_1+\lambda)(A_1+\lambda)\,\beta''^2 + (A_1+\lambda)(B_1+\lambda)\,\gamma''^2 = \frac{1}{\kappa}\,\mathfrak{C}$$

$$(B_1+\lambda)(C_1+\lambda)\,\alpha'\alpha'' + (C_1+\lambda)(A_1+\lambda)\,\beta'\beta'' + (A_1+\lambda)(B_1+\lambda)\,\gamma'\gamma'' = \frac{1}{\kappa}\,\mathfrak{F}$$

$$(B_1+\lambda)(C_1+\lambda)\,\alpha''\alpha + (C_1+\lambda)(A_1+\lambda)\,\beta''\beta + (A_1+\lambda)(B_1+\lambda)\,\gamma''\gamma = \frac{1}{\kappa}\,\mathfrak{G}$$

$$(B_1+\lambda)(C_1+\lambda)\,\alpha\alpha' + (C_1+\lambda)(A_1+\lambda)\,\beta\beta' + (A_1+\lambda)(B_1+\lambda)\,\gamma\gamma' = \frac{1}{\kappa}\,\mathfrak{H}$$

where, writing down the expanded values of $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}$,

$$(\lambda b + B)(\lambda c + C) - (\lambda f + F)^2 = \mathfrak{A},$$
$$(\lambda c + C)(\lambda a + A) - (\lambda g + G)^2 = \mathfrak{B},$$
$$(\lambda a + A)(\lambda b + B) - (\lambda h + H)^2 = \mathfrak{C},$$
$$(\lambda g + G)(\lambda h + H) - (\lambda a + A)(\lambda f + F) = \mathfrak{F},$$
$$(\lambda h + H)(\lambda f + F) - (\lambda b + B)(\lambda g + G) = \mathfrak{G},$$
$$(\lambda f + F)(\lambda g + G) - (\lambda c + C)(\lambda h + H) = \mathfrak{H}.$$

By writing successively $\lambda = -A_1$, $\lambda = -B_1$, $\lambda = -C_1$, we see in the first place that A_1, B_1, C_1 are the roots of the same cubic equation, and we obtain next the values of $\alpha^2, \beta^2, \gamma^2$, &c. in terms of these quantities A_1, B_1, C_1, and of the coefficients a, b, &c., A, B, &c. It is easy to see how the above formulæ would have been modified if a_1, b_1, c_1, instead of being equal to unity, had one or more of them been equal to unity with a negative sign. It is obvious that every step of the preceding process is equally applicable whatever the number of variables.

ON THE ATTRACTION OF AN ELLIPSOID.

By Arthur Cayley.

PART I.—ON LEGENDRE'S SOLUTION OF THE PROBLEM OF THE ATTRACTION OF AN ELLIPSOID ON AN EXTERNAL POINT.

I propose in the following paper to give an outline of Legendre's investigation of the attraction of an ellipsoid upon an exterior point, one of the earliest and (notwithstanding its complexity) most elegant solutions of the problem. It will be convenient to begin by considering some of the geometrical properties of a system of cones made use of in the investigation.

§ 1. The equation of the ellipsoid referred to axes parallel to the principal axes, and passing through the attracted point, may be written under the form

$$l(x-a)^2 + m(y-b)^2 + n(z-c)^2 - k = 0,$$

(where $\sqrt{\dfrac{k}{l}}$, $\sqrt{\dfrac{k}{m}}$, $\sqrt{\dfrac{k}{n}}$ are the semiaxes, and a, b, c are the

coordinates of the attracted point referred to the principal axes). Or putting $la^2 + mb^2 + nc^2 - k = \delta$, this equation becomes

$$lx^2 + my^2 + nz^2 - 2(lax + mby + ncz) + \delta = 0.$$

The cones in question are those which have the same axes and directions of circular section with the cone having its vertex in the attracted point and circumscribed about the ellipsoid. The equation of the system of cones (containing the arbitrary parameter ω) is

$$(lx^2 + my^2 + nz^2)\,\delta - (lax + mby + ncz)^2 + \omega^2(x^2 + y^2 + z^2) = 0\,;$$

or as it may also be written,

$$(\omega^2 + l\delta - l^2a^2)\,x^2 + (\omega^2 + m\delta - m^2b^2)\,y^2 + (\omega^2 + n\delta - n^2c^2)\,z^2$$
$$- 2mnbcyz - 2nlcazx - 2lmabxy = 0.$$

For $\omega = 0$, the cone coincides with the circumscribed cone; as ω increases, the aperture of the cone gradually diminishes, until for a certain value, $\omega = \Omega$, the cone reduces itself to a straight line (the normal of the confocal ellipsoid through the attracted point). It is easily seen that Ω^2 is the positive root of the equation

$$\frac{l^2a^2}{\Omega^2 + l\delta} + \frac{m^2b^2}{\Omega^2 + m\delta} + \frac{n^2c^2}{\Omega^2 + n\delta} = 1,$$

a different form of which may be obtained by writing $\Omega^2 = \dfrac{k\delta}{\xi}$, ξ being then determined by means of the equation

$$\frac{la^2}{k + l\xi} + \frac{mb^2}{k + m\xi} + \frac{nc^2}{k + n\xi} = 1\,;$$

i.e. $\sqrt{\left(\dfrac{k}{l} + \xi\right)}$, $\sqrt{\left(\dfrac{k}{m} + \xi\right)}$, $\sqrt{\left(\dfrac{k}{n} + \xi\right)}$ are the semiaxes of the confocal ellipsoid through the attracted point.

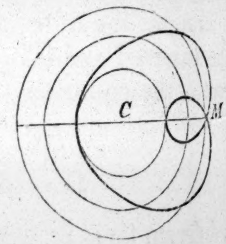
In the case where ω remains indeterminate, it is obvious that the cone intersects the ellipsoid in the curve in which the ellipsoid is intersected by a certain hyperboloid of revolution of two sheets, having the attracted point for a focus, and the plain of contact of the ellipsoid with the circumscribed cone (*i.e.* the polar plane of the attracted point) for the corresponding directrix plane. The excentricity of the hyperboloid is $\dfrac{1}{\omega}\sqrt{(l^2a^2 + m^2b^2 + n^2c^2)}$, which suffices for its complete determination. For $\omega = 0$, the hyperboloid reduces itself

to the plane of contact of the ellipsoid with the circumscribed cone, and for $\omega = \Omega$, the hyperboloid and the ellipsoid have a double contact, viz. at the points where the ellipsoid is intersected by the normal to the confocal ellipsoid through the attracted point.

If ω remains constant while k is supposed to vary, *i.e.* if the ellipsoid vary in magnitude (the position and proportion of its axes remaining unaltered) the locus of the intersection of the cone and the ellipsoid is a surface of the fourth order defined by the equation

$$(lx^2 + my^2 + nz^2 - lax - mby - ncz)^2 = \omega^2 (x^2 + y^2 + z^2),$$

and consisting of an exterior and an interior sheet meeting at the attracted point, which is a conical point on the surface, *i.e.* a point where the tangent plane is replaced by a tangent cone. The general form of this surface is easily seen from the figure, in which the ellipsoid has been replaced by a sphere, and the surface in question is that generated by the revolution of the curve round the line CM. The surface of the fourth order being once described for any particular value of ω, the cone corresponding to any one of the series of similar, similarly situated, and concentric ellipsoids is at once determined by means of the intersection of the ellipsoid in question with the surface of the fourth order. It is clear too that there is always one of these ellipsoids which has a double contact with the surface of the fourth order, viz. at the points where this ellipsoid is intersected by the normal to the confocal ellipsoid through the attracted point; *i.e.* there is always an ellipsoid for which the cone corresponding to a given value of ω reduces itself to a straight line.



Consider the attracting ellipsoid, which for distinction may be termed the ellipsoid S, and the two cones C, C', which correspond to the values ω, $\omega - d\omega$ of the variable parameter. Legendre shews that the attraction of the portion of the ellipsoid S included between the two cones C, C' is independent of the quantity k, which determines the magnitude of the ellipsoid, *i.e.* if there be any other ellipsoid T similarly situated and concentric to and with the ellipsoid S; and two cones D, D', which for the ellipsoid T correspond to the same values ω, $\omega - d\omega$ of the variable parameter, the attraction of the portion of the ellipsoid S included between

the two cones C, C', is equal to the attraction of the portion of the ellipsoid T included between the two cones D and D'. By taking for the ellipsoid T the ellipsoid for which the cone D reduces itself to a straight line, the aperture of the cone D' is indefinitely small, and the attraction of the portion of the ellipsoid T included within the cone D' is at once determined; *i. e.* the attraction of the portion of the ellipsoid S included between the cones C, C' is obtained in a finite form. Hence the attraction of the portion of the ellipsoid S included between any two cones $C_{,}$, $C_{,,}$ corresponding to the values $\omega_{,}$, $\omega_{,,}$ of the variable parameter, is expressed by means of a single integral, and by extending the integration from $\omega = 0$ to $\omega = \sqrt{\left(\frac{k\delta}{\xi}\right)}$, the attraction of the whole ellipsoid is obtained in the form of a single integral readily reducible to that given by the ordinary solutions. It is clear too that the attraction of the portion of the ellipsoid S included between any two cones $C_{,}$, $C_{,,}$, is equal to that of the portion of the ellipsoid T included between the corresponding cones $D_{,}$ and $D_{,,}$. Hence also, assuming for the ellipsoid T, that for which the cone $D_{,,}$ reduces itself to a straight line, and supposing that the cones $C_{,}$ and $D_{,}$ coincide with the circumscribing cones, the attraction of the portion of the ellipsoid S exterior to the cone $C_{,,}$ is equal to the attraction of the entire ellipsoid T. Or more generally, the attraction of the portion of the ellipsoid S included between the cones $C_{,}$ and $C_{,,}$ is equal to the attraction of the shell included between the surfaces of the two ellipsoids, for which the cones $D_{,}$ and $D_{,,}$ respectively reduce themselves to straight lines.

§ 2. Proceeding to the analytical solution, and resuming the equation of the ellipsoid

$$lx^2 + my^2 + nz^2 - 2(lax + mby + ncz) + \delta = 0,$$

and that of the cone

$$(lx^2 + my^2 + nz^2)\,\delta - (lax + mby + ncz)^2 + \omega^2(x^2 + y^2 + z^2) = 0.$$

Consider a radius vector on the conical surface such that the cosines of its inclinations to the axes are

$$\frac{P}{\Theta}, \frac{Q}{\Theta}, \frac{R}{\Theta}, \{\Theta = \sqrt{(P^2 + Q^2 + R^2)}\},$$

P, Q, R and Θ being functions of the parameter ω, and of another variable ϕ, which determines the position of the radius vector upon the conical surface. Also let ρ be the

length of the portion of the radius vector which lies within the ellipsoid; then representing by dS the spherical angle corresponding to the variations of ω and ϕ, the attraction in the direction of the axis of x is given by the formula

$$A = \iint \rho \frac{P}{\Theta} \, dS.$$

Also by a known formula

$$dS = \frac{1}{\Theta^3} \left\{ P \left(\frac{dQ}{d\phi} \frac{dR}{d\omega} - \frac{dR}{d\phi} \frac{dQ}{d\omega} \right) + Q \left(\frac{dR}{d\phi} \frac{dP}{d\omega} - \frac{dR}{d\omega} \frac{dP}{d\phi} \right) + R \left(\frac{dP}{d\phi} \frac{dQ}{d\omega} - \frac{dP}{d\omega} \frac{dQ}{d\phi} \right) \right\},$$

and it is easy to obtain

$$\rho = \frac{2\omega\Theta^2}{lP^2 + mQ^2 + nR^2}.$$

The quantities P, Q, R have now to be expressed as functions of ω, ϕ, so that their values substituted for x, y, z, may satisfy identically the equation of the cone. This may be done by assuming

$$P = p,$$

$$Q = mb\,(\omega^2 + n\delta) \left(la + \frac{D}{U} \cos \phi \right) - \frac{D\sqrt{p}}{U} nc \sin \phi,$$

$$R = nc\,(\omega^2 + m\delta) \left(la + \frac{D}{U} \cos \phi \right) + \frac{D\sqrt{p}}{U} mb \sin \phi,$$

where $p = (\omega^2 + m\delta)(\omega^2 + n\delta) - m^2b^2(\omega^2 + n\delta) - n^2c^2(\omega^2 + m\delta)$,

$$U^2 = m^2b^2(\omega^2 + n\delta) + n^2c^2(\omega^2 + m\delta),$$

$$D^2 = (\omega^2 + l\delta)(\omega^2 + m\delta)(\omega^2 + n\delta) \left\{ \frac{l^2a^2}{\omega^2 + l\delta} + \frac{m^2b^2}{\omega^2 + m\delta} + \frac{n^2c^2}{\omega^2 + n\delta} - 1 \right\}.$$

A system of values which, in point of fact, depend upon the following geometrical considerations: by treating x as a constant in the equation of the cone, *i.e.* in effect by considering the sections of the cone by planes parallel to that of yz, the equation of the cone becomes that of an ellipse; transforming first to a set of axes through the centre and then to a set of conjugate axes, one of which passes through the point where the plane of the ellipse is intersected by the axis of x, the equation takes the form $\frac{\xi^2}{A^2} + \frac{\eta^2}{B^2} = 1$, and is satisfied by $\xi = A \cos \phi$, $\eta = B \sin \phi$, and $\frac{y}{x}$, $\frac{z}{x}$ being of course linear

functions of these values, the preceding expressions may be obtained.

The substitution of the above values of P, Q, R (a somewhat tedious one which does not occur in the process actually made use of by Legendre) gives the very simple result,

$$dS = \frac{P^{\frac{1}{2}}\omega\, d\omega\, d\phi}{\Theta}.$$

And the formula for the attraction becomes

$$A = 2\iint \frac{P^{\frac{3}{2}}\,\omega^2 d\omega d\phi}{lP^2 + mQ^2 + nR^2};$$

which is of the form $A = 2\int I\omega^2 d\omega$, where

$$I = \int \frac{P^{\frac{3}{2}} d\phi}{lP^2 + mQ^2 + nR^2},$$

which last integral, taken between the limits $\phi = 0$ and $\phi = 2\pi$, and multiplied by $2\omega^2 d\omega$, expresses the attraction of the portion of the ellipsoid included between two consecutive cones. The integration is evidently possible, but the actual performance of it is the great difficulty of Legendre's process. The result, as before mentioned, is independent of the quantity k, or, what comes to the same thing, of the quantity δ: assuming this property (an assumption which in fact resolves itself into the consideration of the ellipsoid for which the cone reduces itself to a straight line, as before explained), the integral is at once obtained by writing $\delta = \Delta$ where Δ represents the positive root of the equation

$$\frac{l^2a^2}{\omega^2 + l\Delta} + \frac{m^2b^2}{\omega^2 + m\Delta} + \frac{n^2c^2}{\omega^2 + n\Delta} - 1 = 0.$$

This gives
$$P = \frac{l^2a^2(\omega^2 + m\Delta)(\omega^2 + n\Delta)}{\omega^2 + l\Delta},$$

$$Q = lmab(\omega^2 + n\Delta),$$

$$R = lnac(\omega^2 + m\Delta),$$

values independent of ϕ, or the value of I is found by multiplying the quantity under the integral sign by 2π: and hence we have

$$A = 4\pi la \times$$

$$\int \frac{\omega^2(\omega^2 + l\Delta)^{\frac{1}{2}}(\omega^2 + m\Delta)^{\frac{3}{2}}(\omega^2 + n\Delta)^{\frac{3}{2}}\,d\omega}{l^3a^2(\omega^2+m\Delta)^2(\omega^2+n\Delta)^2+m^3b^2(\omega^2+n\Delta)^2(\omega^2+l\Delta)^2+n^3c^2(\omega^2+n\Delta)^2(\omega^2+l\Delta)^2},$$

where of course Δ is to be considered as a function of ω. By integrating from $\omega = \omega_{,}$ to $\omega = \omega_{,,}$, we have the attraction of the portion of the ellipsoid included between any two of the series of cones, and to obtain the attraction of the whole ellipsoid we must integrate from $\omega = 0$ to $\omega = \sqrt{\left(\frac{k\delta}{\xi}\right)}$, where ξ is determined as before by the equation

$$\frac{la^2}{k + l\xi} + \frac{mb^2}{k + m\xi} + \frac{nc^2}{k + n\xi} = 1.$$

And it is obvious that for this value of ω we have $\Delta = \delta$. The expression for the attraction is easily reduced to a known form by writing $y = \frac{k\Delta}{\omega^2}$; this gives

$$A = 4\pi la \times$$

$$\int \frac{k^{\frac{1}{2}}\omega d\omega\,(k+ly)^{\frac{1}{2}}(k+my)^{\frac{3}{2}}(k+ny)^{\frac{3}{2}}}{l^3a^2(k+my)^2(k+ny)^2 + m^3b^2(k+ny)^2(k+ly)^2 + n^3c^2(k+ly)^2(k+my)^2}.$$

Also $$\omega^2 = k\left(\frac{l^2a^2}{k+ly} + \frac{m^2b^2}{k+my} + \frac{n^2c^2}{k+ny}\right);$$

whence $\omega d\omega = -$

$$\frac{k[l^3a^2(k+my)^2(k+ny)^2 + m^3b^2(k+ny)^2(k+ly)^2 + n^3c^2(k+ly)^2(k+my)^2]}{2\,(k+ly)^2(k+my)^2(k+ny)^2},$$

or $$A = 2\pi k^{\frac{3}{2}} la \int \frac{dy}{(k+ly)^{\frac{3}{2}}(k+my)^{\frac{1}{2}}(k+ny)^{\frac{1}{2}}},$$

where for the entire ellipsoid the integral is to be taken from $y = \xi$ to $y = \infty$. A better known form is readily obtained by writing $x^2 = \frac{k + l\xi}{k + ly}$, in which case the limits for the entire ellipsoid are $x = 0$, $x = 1$.

It may be as well to indicate the first step of the reduction of the integral I, viz. the method of resolving the denominator into two factors. We have identically,

$$(\Delta - \delta)(lP^2 + mQ^2 + nR^2)$$

$$= \omega^2(P^2 + Q^2 + R^2) + \Delta\,(lP^2 + mQ^2 + nR^2) - (laP + mbQ + ncR)^2.$$

And the second side of this equation is resolvable into two factors independently of the particular values of P, Q, R. Representing this second side for a moment in the notation of a general quadratic function, or under the form

$$AP^2 + BQ^2 + CR^2 + 2FQR + 2GRP + 2HPQ,$$

we have the required resolution,

$$lP^2 + mQ^2 + nR^2 = \frac{1}{A}[AP + \{H + \sqrt{(-\mathfrak{C})}\}\,Q + \{G + \sqrt{(-\mathfrak{B})}\}\,R]$$

$$[AP + \{H - \sqrt{(-\mathfrak{C})}\}\,Q + \{G - \sqrt{(-\mathfrak{B})}\}\,R];$$

where, as usual, $\mathfrak{B} = CA - G^2$, $\mathfrak{C} = AB - H^2$, and the roots must be so taken that $\sqrt{(-\mathfrak{B})}\sqrt{(-\mathfrak{C})} = \mathfrak{F}$ $\{\mathfrak{F} = (GH - AF)\}$.

I have purposely restricted myself so far to the problem considered by Legendre: the general transformation, of which the preceding is a particular case, and also a simpler mode of effecting the integration, are given in the next part of this paper.

PART II.—ON A FORMULA FOR THE TRANSFORMATION OF CERTAIN MULTIPLE INTEGRALS.

Consider the integral

$$V = \int F(x, y, \ldots)\, dx\, dy,$$

where the number of variables $x, y, \ldots$ is equal to n, and $F(x, y \ldots)$ is a homogeneous function of the order μ.

Suppose that $x, y, \ldots$ are connected by a homogeneous equation $\psi(x, y, \ldots) = 0$ containing a variable parameter ω (so that ω is a homogeneous function of the order zero in the variables $x, y, \ldots$). Then, writing

$$r^2 = x^2 + y^2 + \ldots$$
$$x = r\alpha,\ y = r\beta \ldots$$

the quantities $\alpha, \beta, \ldots$ are connected by the equations

$$\alpha^2 + \beta^2 + \ldots = 1,$$
$$\psi(\alpha, \beta, \ldots) = 0,$$

and we may therefore consider them as functions of ω and of $(n-2)$ independent variables θ, ϕ, &c.; whence

$$dx\, dy \ldots = r^{n-1} \nabla\, dr\, d\omega\, d\theta,$$

where

$$\nabla = \begin{vmatrix} \alpha, & \beta, & \ldots \\ \dfrac{d\alpha}{d\omega}, & \dfrac{d\beta}{d\omega}, & \\ \dfrac{d\alpha}{d\theta}, & \dfrac{d\beta}{d\theta}, & \end{vmatrix}$$

Also $$F(x, y, \ldots) = r^{\mu} F(a, \beta \ldots),$$

and therefore $V = \int r^{\mu+n-1} F(a, \beta, \ldots) \nabla \, dr \, d\omega \, d\theta.$

Or, integrating with respect to r,

$$\int r^{\mu+n-1} \, dr = \frac{1}{\mu+n} r^{\mu+n},$$

which, taken between the proper limits, is a function of $a, \beta, \ldots$ equal $f(a, \beta, \ldots)$ suppose; this gives

$$V = \int f(a, \beta, \ldots) F(a, \beta, \ldots) \nabla \, d\omega \, d\theta \ldots,$$

in which I shall assume that the limits of ω are constant. If, in order to get rid of the condition $a^2 + \beta^2, \ldots = 1$, we assume

$$a = \frac{p}{\rho}, \quad p = \frac{q}{\rho}, \ldots$$

$$\rho^2 = p^2 + q^2 + \ldots$$

The preceding expression for V becomes

$$V = \int f\left(\frac{p}{\rho}, \frac{q}{\rho}, \ldots\right) F(p, q, \ldots) \frac{1}{\rho^{\mu+n}} D \, d\omega \, d\theta \ldots$$

in which $$D = \begin{vmatrix} p, & q, & \ldots \\ \frac{dp}{d\omega}, & \frac{dq}{d\omega}, & \\ \frac{dp}{d\theta}, & \frac{dq}{d\theta}. & \end{vmatrix}$$

Assume $$p = P\xi + P'\eta + P''\zeta \ldots$$
$$q = Q\xi + Q'\eta + Q''\zeta \ldots$$
$$\vdots$$

where the number of variables $\xi, \eta, \zeta \ldots$ (functions in general of ω, θ, &c.) is n, and where the coefficients P, Q, &c. are supposed to be functions of ω only. We have

$$\frac{dp}{d\theta} = P \frac{d\xi}{d\theta} + P' \frac{d\eta}{d\theta} + P'' \frac{d\zeta}{d\theta} \ldots$$

$$\frac{dq}{d\theta} = Q \frac{d\xi}{d\theta} + Q' \frac{d\eta}{d\theta} + Q'' \frac{d\zeta}{d\theta} \ldots$$

And, substituting these values as well of those of p, q, &c., but retaining the terms $\frac{dp}{d\omega}$, $\frac{dq}{d\omega}$, &c. in their original form.

The determinant D resolves itself into the sum of a series of products,

$$\begin{vmatrix} \frac{dp}{d\omega}, & \frac{dq}{d\omega}, & \ldots \\ P', & Q' & \\ P'', & Q'' & \\ \vdots & & \end{vmatrix} \begin{vmatrix} 1 & . & \ldots \\ . & \eta, & \zeta \\ . & \frac{d\eta}{d\theta}, & \frac{d\zeta}{d\theta} \\ \vdots & & \end{vmatrix}$$

Let Ψ be the function to which $\psi(p, q, \ldots)$ is changed by the substitution of the above values of $p, q, \ldots$ Ψ is a homogeneous function of $\xi, \eta, \zeta, \ldots$ and we have the relation $\Psi = 0$. (It will be convenient to consider $\xi, \eta, \zeta, \ldots$ as functions of ω, θ, &c., such as to satisfy identically this last equation). We deduce

$$\frac{1}{X}\begin{vmatrix} 1, & . & . \\ . & \eta, & \zeta, \ldots \\ . & \frac{d\eta}{d\theta}, & \frac{d\zeta}{d\theta} \\ \vdots & & \end{vmatrix} = \frac{1}{Y}\begin{vmatrix} . & 1, & . \\ \xi, & . & \zeta, \ldots \\ \frac{d\xi}{d\theta}, & . & \frac{d\zeta}{d\theta} \\ \vdots & & \end{vmatrix} = \text{\&c.} = S \text{ suppose,}$$

where for shortness $X = \frac{d\Psi}{d\xi}$, $Y = \frac{d\Psi}{d\eta}$, &c. The substitution of these values gives

$$D = \begin{vmatrix} & \frac{dp}{d\omega}, & \frac{dq}{d\omega}, & \ldots \\ X, & P, & Q, & \\ Y, & P', & Q', & \\ Z, & P'', & Q'', & \\ \vdots & & & \end{vmatrix} S,$$

where it will be remarked that the successsve horizontal lines (after the first) of the determinant are the differential coefficients of $\Psi, p, q, \ldots$ with respect to ξ with respect to η, &c. In general, if ψ denote any function of $p, q, \ldots$ these quantities being themselves functions of $\omega, \xi, \eta, \ldots$ and $\xi, \eta, \ldots$ containing ω, also if Ψ be what ψ becomes when for $p, q, \ldots$ we substitute their values in $\omega, \xi, \eta, \ldots$ we have identically

$$\begin{vmatrix} \frac{d\psi}{d\omega}, & \frac{dp}{d\omega}, & \frac{dq}{d\omega}, \dots \\ X, & P, & Q, \\ Y, & P', & Q', \\ Z, & P'', & Q'', \\ \vdots & & \end{vmatrix} = 0^*.$$

In the present case however, writing for shortness $\psi(p, q, ..)$ $= \psi$, this function ψ contains ω explicitly as well as implicitly through p, q, &c. The formula is still true if for $\frac{d\psi}{d\omega}$ we substitute $\frac{d(\psi)}{d\omega} - \frac{d\psi}{d\omega}$, $\frac{d\psi}{d\omega}$ on the second side, denoting a partial differential coefficient taken only so far as ω is explicitly contained in ψ. And considering $p, q, ..$ as functions of $\omega, \xi, \eta, ..$ ($\xi, \eta, ..$ themselves functions of ω and of other variables which need not here be considered), ψ or Ψ vanishes identically, and we have $\frac{d(\psi)}{d\omega} = 0$. Hence, in the last formula, we have to write $-\frac{d\psi}{d\omega}$ instead of $\frac{d\psi}{d\omega}$, and we thus derive

$$\frac{d\psi}{d\omega}\begin{vmatrix} P, & Q, \dots \\ P', & Q', \\ \vdots & \end{vmatrix} = \begin{vmatrix} \cdot & \frac{dp}{d\omega}, & \frac{dq}{d\omega}, \dots \\ X, & P, & Q, \\ Y, & P', & Q', \\ Z, & P'', & Q'', \\ \vdots & & \end{vmatrix},$$

whence D becomes

$$D = \begin{vmatrix} P, & Q \dots \\ P', & Q' \\ \vdots & \end{vmatrix} \frac{d\psi}{d\omega} S,$$

which is the value to be made use of in the equation

$$V = \int f\left(\frac{p}{\rho}, \frac{q}{\rho} \dots\right) F(p, q, \dots) \frac{1}{\rho^{\mu+n}} D\, d\omega\, d\theta \dots$$

The principal use of the formula is where ψ is a homogeneous

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* This formula, or one equivalent to it, is given in Jacobi's memoir "De Determinantibus Functionalibus."—*Crelle*, tom. XXII. p. 319.

function of the second order of p, q,.. Thus, suppose

$$\psi = \tfrac{1}{2}(Ap^2 + Bq^2 \ldots + 2Hpq + \ldots).$$

Also, for the sake of conformity to the usual notation in the theory of transformation of quadratic functions, writing $\alpha, \alpha', \ldots \beta, \beta' \ldots$ instead of $P, P', \ldots Q, Q', \ldots$ and putting after the differentiations $\xi = 1$, we have

$$\begin{aligned} p &= \alpha + \alpha'\eta + \alpha''\zeta \ldots \\ q &= \beta + \beta'\eta + \beta''\zeta \ldots, \\ &\vdots \end{aligned}$$

values which we may assume to give rise to the equation

$$(Ap^2 + Bq^2 \ldots + 2Hpq \ldots) = (1 - \eta^2 - \zeta^2 \ldots),$$

(which supposes η, ζ,.. to be functions of θ, &c. such as to satisfy identically the equation $1 = \eta^2 + \zeta^2 + \ldots$).

Hence, by a well-known property, if

$$\begin{vmatrix} A, & H & \ldots \\ H, & B & \\ \vdots & & \end{vmatrix} = \kappa,$$

we have

$$\begin{vmatrix} \alpha, & \beta & \ldots \\ \alpha', & \beta' & \\ \vdots & & \end{vmatrix} = \sqrt{\left\{\frac{(-)^{n-1}}{\kappa}\right\}};$$

so that, observing that in the present case $X = 1$, and therefore

$$S = \begin{vmatrix} \eta, & \zeta & \ldots \\ \dfrac{d\eta}{d\theta}, & \dfrac{d\zeta}{d\theta} & \\ \vdots & & \end{vmatrix}$$

we have

$$V = \int \frac{(-)^{\frac{1}{2}(n-1)}}{\kappa^{\frac{1}{2}}} f\left(\frac{p}{\rho}, \frac{q}{\rho} \ldots\right) F(p, q \ldots) \frac{1}{\rho^{\mu+n}} \frac{d\psi}{d\omega} S \, d\omega \, d\theta \ldots$$

The remainder of the process of integration may in many cases be effected by the method made use of by Jacobi in the memoir "De binis quibuslibet functionibus, &c." *Crelle*, tom. XII. p. 1, viz. the coefficients α, α', &c., β, &c., may in addition to the conditions which they are already supposed to satisfy, be so determined as to reduce any homogeneous function of p, q, r,.. entering into the integral to a form containing the squares only of the variables. This method is applied in the memoir in question to the integrals of n

variables, analogous to those which give the attraction of an ellipsoid; and that directly without effecting an integration with respect to the radius vector. I proceed to shew how the preceding investigations lead to Legendre's integral, and how the method in question effects with the utmost simplicity the integration which Legendre accomplished by means of what Poisson has spoken of as inextricable calculations.

Consider in particular the formula

$$V = \int \frac{(x, y\ldots)^h dx\, dy\ldots}{(x^2 + y^2\ldots)^{\frac{3}{2}-i}},$$

the number of variables being as before n, and $(x, y\ldots)^h$ denoting a homogeneous function of the order h. The equation for the limits is assumed to be

$$l(x-a)^2 + m(y-b)^2 + \ldots\ldots = k.$$

Assume

$$\psi(x, y\ldots) = \omega^2(x^2 + y^2\ldots) + \delta(lx^2 + my^2 + \ldots) - (lax + mby + \ldots)^2,$$

(where $\delta = la^2 + mb^2\ldots - k$),* or more simply,

$$\psi(x, y\ldots) = (\omega^2 + l\delta - l^2a^2)x^2 + (\omega^2 + m\delta - m^2b^2)y^2 + \ldots - 2lmab\,xy - \ldots$$

Here $\mu = h + 2i - 3$. Also, putting for shortness

$$lap + mbq + \ldots\ldots = \Lambda,\quad lp^2 + mq^2 + \ldots\ldots = \Phi.$$

It is easy to obtain

$$f\left(\frac{p}{\rho}, \frac{q}{\rho}\ldots\right) = \frac{1}{h+2i+n-3}\,\frac{\rho^{h+2i+n-3}\left[(\Lambda+\omega\rho)^{h+2i+n-3} - (\Lambda-\omega\rho)^{h+2i+n-3}\right]}{\Phi^{h+2i+n-3}}.$$

Also $\quad F(p, q\ldots) = \dfrac{(p, q\ldots)^h}{\rho^{3-2i}},\quad \dfrac{d\psi}{d\omega} = 2\omega\rho^2,$

$$\kappa = (\omega^2 + l\delta)(\omega^2 + m\delta)\ldots\left(1 - \frac{l^2a^2}{\omega^2 + l\delta} - \frac{m^2b^2}{\omega^2 + m\delta} - \&c.\right),$$

values which give

$$V = \frac{2}{h + 2i + n - 3}$$

$$\int \frac{(-)^{\frac{1}{2}(n-1)}}{\kappa^{\frac{1}{2}}}\,\frac{\omega\rho^{2k-1}\left[(\Lambda+\omega\rho)^{h+2i+n-3} - (\Lambda-\omega\rho)^{h+2i+n-3}\right](p, q\ldots)^h S\,d\omega\,d\theta\ldots}{\Phi^{h+2i+n-3}},$$

where it will be remembered that $p, q\ldots$ are linear functions (with constant terms) of $(n-1)$ variables $\eta, \zeta\ldots$, these last mentioned quantities being themselves functions of $(n-2)$ variables θ, &c. such that $1 - \eta^2 - \zeta^2\ldots\ldots = 0$ identically. If

* It is tacitly assumed in the sequel that δ is positive.

besides we suppose that $\Phi = lp^2 + mq^2 \ldots$ reduces itself to the form $\frac{1}{P} - \frac{\eta^2}{Q} - \&c.$, we have, by the formula of the paper "On the Simultaneous Transformation of two Homogeneous Equations of the Second Order,"

$$\left(1 - \frac{\lambda}{P}\right)\left(1 - \frac{\lambda}{Q}\right) \ldots = \frac{1}{\kappa}(\omega^2 + l\delta - l\lambda)(\omega^2 + m\delta - m\lambda) \ldots \times$$

$$\left(1 - \frac{l^2a^2}{\omega^2 + l\delta - l\lambda} - \frac{m^2b^2}{\omega^2 + m\delta - m\lambda} - \ldots\right),$$

which is true, whatever be the value of λ.

It seems difficult to proceed further with the general formula, and I shall suppose $n = 3$, $i = 0$, $h = 1$, $(x, y \ldots)^h = x$, or

$$V = \int \frac{x\,dx\,dy\,dz}{(x^2 + y^2 + z^2)^{\frac{3}{2}}},$$

the equation of the limits being

$$l(x - a)^2 + m(y - b)^2 + n(z - c)^2 = k.$$

Here we may assume $\eta = \cos\theta$, $\zeta = \sin\theta$, (values which give $S = 1$). And we have

$$V = 2\int \frac{\omega^2\,d\omega}{\kappa^{\frac{1}{2}}} \int \frac{(a + a'\cos\theta + a''\sin\theta)\,d\theta}{\frac{1}{P} - \frac{\cos^2\theta}{Q} - \frac{\sin^2\theta}{R}},$$

from $\theta = 0$ to $\theta = 2\pi$. Or, what comes to the same thing,

$$V = 8\int \frac{\omega^2\,d\omega}{\kappa^{\frac{1}{2}}} \int \frac{a\,d\theta}{\frac{1}{P} - \frac{\cos^2\theta}{Q} - \frac{\sin^2\theta}{R}},$$

from $\theta = 0$ to $\theta = \frac{1}{2}\pi$. Whence

$$V = 4\pi \int \frac{a\omega^2 P\sqrt{(QR)}\,d\omega}{\kappa^{\frac{1}{2}}\sqrt{\{(P - Q)(P - R)\}}},$$

we have from the formulæ of the paper before quoted,

$$a^2 = \frac{QR}{\kappa}\,\frac{(B - mP)(C - nP) - F^2}{(P - Q)(P - R)},$$

B, C, F, being the coefficients of y^2, z^2, yz in $\psi(x, y, z)$, viz.

$$B = \omega^2 + m\delta - m^2b^2, \quad C = \omega^2 + n\delta - n^2c^2, \quad F = mnbc.$$

Whence
$$V = 4\pi \int \omega^2\, d\omega\, \frac{PQR}{\kappa}\, \frac{\{(B - mP)(C - nP) - F^2\}^{\frac{1}{2}}}{(P - Q)(P - R)}.$$

Also from the equation

$$\left(1 - \frac{\lambda}{P}\right)\left(1 - \frac{\lambda}{Q}\right)\left(1 - \frac{\lambda}{R}\right) = \frac{1}{\kappa}(\omega^2 + l\delta - l\lambda)(\omega^2 + m\delta - m\lambda)(\omega^2 + n\delta - n\lambda)\left(1 - \frac{l^2a^2}{\omega^2 + l\delta - l\lambda} - \frac{m^2b^2}{\omega^2 + m\delta - m\lambda} - \frac{n^2c^2}{\omega^2 + n\delta - n\lambda}\right);$$

differentiating with respect to λ, and writing $\lambda = P$,

$$\frac{-\dfrac{1}{P}\left(1 - \dfrac{P}{Q}\right)\left(1 - \dfrac{P}{R}\right)}{(\omega^2 + l\delta - lP)(\omega^2 + m\delta - mP)(\omega^2 + n\delta - nP)} = -\frac{1}{\kappa}\left\{\frac{l^3a^2}{(\omega^2 + l\delta - lP)^2} + \frac{m^3b^2}{(\omega^2 + m\delta - mP)^2} + \frac{n^3c^2}{(\omega^2 + n\delta - nP)^2}\right\},$$

or
$$\frac{PQR}{\kappa(P - Q)(P - R)} = \frac{1}{(\omega^2 + l\delta - lP)(\omega^2 + m\delta - mP)(\omega^2 + n\delta - nP)\left\{\dfrac{l^3a^2}{(\omega^2 + l\delta - lP)^2} + \dfrac{m^3b^2}{(\omega^2 + m\delta - mP)^2} + \dfrac{n^3c^2}{(\omega^2 + n\delta - nP)^2}\right\}},$$

and from the values first written down, for B, C, F, —— $(B - mP)(C - nP) - F^2$

$$= (\omega^2 + m\delta)(\omega^2 + n\delta) - m^2b^2(\omega^2 + n\delta) - n^2c^2(\omega^2 + m\delta) - mP(\omega^2 + n\delta - n^2c^2) - nP(\omega^2 + m\delta - m^2b^2) + mnP^2,$$

$$= (\omega^2 + m\delta - mP)(\omega^2 + n\delta - nP) - m^2b^2(\omega^2 + n\delta - nP) - n^2c^2(\omega^2 + m\delta - mP),$$

$$= \frac{l^2a^2(\omega^2 + m\delta - mP)(\omega^2 + n\delta - nP)}{\omega^2 + l\delta - lP},$$

the last reduction being effected by means of the equation

$$1 - \frac{l^2a^2}{\omega^2 + l\delta - lP} - \frac{m^2b^2}{\omega^2 + m\delta - mP} - \frac{n^2c^2}{\omega^2 + n\delta - nP} = 0.$$

Hence $$\frac{PQR}{\kappa}\,\frac{\{(B - mP)(C - nP) - F^2\}^{\frac{1}{2}}}{(P - Q)(P - R)} =$$

$$\frac{la}{(\omega^2 + l\delta - lP)^{\frac{3}{2}}(\omega^2 + m\delta - mP)^{\frac{1}{2}}(\omega^2 + n\delta - nP)^{\frac{1}{2}}\left\{\frac{l^3a^2}{(\omega^2 + l\delta - lP)^2} + \frac{m^3b^2}{(\omega^2 + m\delta - mP)^2} + \frac{n^3c^2}{(\omega^2 + n\delta - nP)^2}\right\}}.$$

And substituting this value, and multiplying so as to get rid of the fractions in the denominator,

$$V = 4\pi la \int \frac{\omega^2(\omega^2 + l\delta - lP)^{\frac{1}{2}}(\omega^2 + m\delta - mP)^{\frac{3}{2}}(\omega^2 + n\delta - nP)^{\frac{3}{2}}\,d\omega}{l^3a^2(\omega^2+m\delta-mP)^2(\omega^2+n\delta-nP)^2+m^3b^2(\omega^2+n\delta-nP)^2(\omega^2+l\delta-lP)^2+n^3c^2(\omega^2+l\delta-lP)^2(\omega^2+m\delta-mP)^2},$$

the reduction of which integral has been already treated of in the former part of this present memoir.

ON A CLASS OF CURVES ON THE HYPERBOLOID OF ONE SHEET CONNECTED WITH THE GENERATRICES OF THE SURFACE.

By R. Townsend. F. T. C. D.

Taking any central surface of the second order and a concentric sphere of arbitrary radius, and considering the nature of the curve locus of their common points, and also of the developable surface envelope of their common tangent planes, we see without any difficulty, that, in the case of the ellipsoid, the curve and developable will be always either both together real or both together imaginary; in the case of the hyperboloid of two sheets, that always one will be real and the other imaginary; and in the case of the hyperboloid of one sheet, that both may be at the same time real, but that they can never be both together imaginary.

Subtracting, one from the other, the equations of the surface and sphere, we get the equation of the cone of the second order diverging from the common center and passing through the common intersecting curve: and substituting, in the equation $p^2 = a^2 \cos^2 \alpha + b^2 \cos^2 \beta + c^2 \cos^2 \gamma$, for the coefficient of p^2 its equivalent $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma$, and then changing the cosines into the corresponding coordinates to which they are proportinal, we get the equation of the cone of central perpendiculars upon all the tangent planes to the common circumscribing developable surface. The equations thus found are, respectively,

$$\left(\frac{1}{a^2} - \frac{1}{r^2}\right) x^2 + \left(\frac{1}{b^2} - \frac{1}{r^2}\right) y^2 + \left(\frac{1}{c^2} - \frac{1}{r^2}\right) z^2 = 0 \ldots (1),$$

$$(a^2 - p^2)\, x^2 + (b^2 - p^2)\, y^2 + (c^2 - p^2)\, z^2 = 0 \ldots (2);$$

from which we see that the former cone, whatever be the value of r, is always concyclic with the cone asymptotic to the given surface, and the cone reciprocal to the latter, whatever be the value of p, is always confocal with the same asymptotic cone; and that, of course, whether these cones be all three real, or whether they be any or all of them imaginary.

Adding together the squared central perpendiculars upon any three rectangular tangent planes to any central surface of the second order, we get the square of the distance from the center of the common intersecting point of these three planes = a constant = $a^2 + b^2 + c^2$; the locus, therefore, of all the intersections of every such system of three tangent planes is always a sphere concentric with the surface. And again, adding together the squared reciprocals of any three rect-

angular central radii of the same surface, we get the squared reciprocal of the central perpendicular let fall upon the planes passing through the extremities of these three radii = another constant $= \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$; the envelope, therefore, of all the planes passing through the extremities of every such system of three central radii is also always a sphere concentric with the given surface: these two particular spheres we shall in what follows denote respectively by the symbols r and $\frac{1}{r}$, and for both the proper signs distinctive of the particular species of central surface are of course in all cases to be attributed to b^2 and c^2.

With respect to the nature of these two spheres and to their relations with the surface, it is obvious that in the case of the ellipsoid they will be always both real, and that they will neither of them ever have either a point or a tangent plane common with that surface; but, on the contrary, in the cases of the two hyperboloids, that they may be either or both of them imaginary; that in the case of the hyperboloid of two sheets, the former sphere, real or imaginary, will never intersect the surface, while, on the contrary, the latter, when real, will always intersect it; and that in the case of the hyperboloid of one sheet, their two curves of intersection common with the surface, and also their two circumscribing developable surfaces in common with themselves enveloping the surface, will each pair be always either both together real or both together imaginary, and that therefore, in the case of that surface, whenever the common intersecting curve and the common circumscribing developable surface are, as they may be, both real for either of the spheres, then always will both be at the same time real for the other sphere also.

The system of cones of the second order which from all the points of the sphere r envelope the surface, whatever be its species, and the system of cones which, diverging from the centre, subtend all the sections of the surface by planes tangent to the sphere $\frac{1}{r}$, enjoy each a peculiar and restrictive property; for, from every point of the former sphere there may be drawn (from the nature of that locus) an infinite number of systems of three rectangular planes, tangents to the given surface, and therefore to the cone which from the same point as vertex envelopes that surface. Hence that cone admits always of an infinite number of systems of

three rectangular tangent planes, and is therefore always itself the reciprocal of the cone of the second order locus of the lines of intersection of its pairs of rectangular tangent planes. And again, to every section of the given surface by a tangent plane to the latter sphere, there may be drawn (from the nature of that envelope) an infinite number of systems of three rectangular central radii; that is, of rectangular sides to the cone which from the centre as vertex subtends that section. Hence that cone admits always of an infinite number of systems of three rectangular sides, and is therefore always itself the reciprocal of the cone of the second order envelope of the system of planes which intersect it each in a pair of rectangular sides.

In the general case of any cone whatever of the second order, it will be presently shewn that the cone, real or imaginary, locus of the lines of intersection of its pairs of rectangular tangent planes is always such that its reciprocal is confocal with the original cone, and that the cone, real or imaginary, envelope of the system of planes which intersect it in pairs of rectangular sides is always such that its reciprocal is concyclic with the original cone. The peculiarity therefore in the two systems of cones just noticed is, that the pairs of cones of the second order, which in the former case are in general merely confocal, and the pairs which in the latter case are in general merely concyclic, are for these two particular systems in both cases actually identical.

Taking now the particular system of points on the sphere r in which that surface intersects the hyperboloid of one sheet, for which alone of the three species of central surface that curve of intersection is ever real: at all these points the enveloping cone becomes infinitely flat, being either portion indifferently of the tangent plane to the hyperboloid bounded by the two rectilinear generatrices in which that plane intersects the surface; the whole system of tangent planes to the surface pass all through one or other of these two generatrices, and thus group themselves into two distinct systems, the cone locus of their assemblage of lines of intersection in pairs at right angles to each other opens out into the two particular tangent planes passing through the normal to the surface at the point on the curve, these two particular tangent planes, together with the plane tangent at the point itself and therefore containing the two generatrices, form a particular system of three tangent planes to the surface at right angles to each other, and those two generatrices themselves consequently intersect at right angles.

Hence the curve, real or imaginary, on the hyperboloid of one sheet locus of the whole system of points for which the two rectilinear generatrices intersect at right angles, that is, for which the two principal curvatures of the surface are equal and opposite, is always a sphero conic, the intersection with the surface of the concentric sphere $r^2 = a^2 + b^2 - c^2$.

Taking, in the next place, the particular system of tangent planes to the sphere $\frac{1}{r}$, which are also tangent planes to the hyperboloid of one sheet, the only one again of the three species of central surface for which that system of common tangent planes is ever real: the conics in which all these planes intersect the surface break up all into pairs of right lines; the rectilinear generatrices of the surface, which two and two intersect at their respective points of contact, and therefore all the cones which from the centre subtend those particular plane sections, open out all into pairs of planes; but, like all other central cones which subtend sections whose planes touch that sphere $\frac{1}{r}$, they all admit each of an infinite number of systems of three rectangular sides, therefore the different pairs of planes forming the whole system are all at right angles to each other. Hence the curve, real or imaginary, on the hyperboloid of one sheet locus of the whole system of points, at which the two rectilinear generatrices viewed from the center of the surface appear to intersect at right angles, is always its line of contact with the developable surface circumscribed in common to itself and to the concentric sphere $\frac{1}{r^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{1}{c^2}$, and is therefore always on a cone of the second order concentric and coaxal with the surface itself.

Again, taking the particular system of points on the sphere r, for which the tangent planes to that surface are also tangent planes to either of the two hyperboloids, both species in this case admitting of a real system of such common tangent planes: at all those points the two tangent planes to the hyperboloid which intersect in the radii of the sphere must, from the nature of that locus, be always at right angles to each other; but those particular tangent planes, as passing through the centre, are all tangent planes to the asymptotic cone of the surface, and the particular system of radii to the sphere in which these tangent planes thus two and two intersect at right angles, we have seen, forms always a cone of the

second order, of which the reciprocal is always confocal with the same asymptotic cone. Hence we see that the assemblage of the lines of intersection of all the pairs of rectangular tangent planes to any cone of the second order forms always another cone of the second order, of which the reciprocal is always confocal with the original cone.

Finally, taking the particular system of points on the sphere $\frac{1}{r}$, in which that surface intersects either of the two hyperboloids, both species again in this case admitting of a real curve of intersection: at all those points the sections of the surface by the tangent planes to the sphere must, from the nature of that envelope, be all equilateral hyperbolas; for, the particular central radii of the hyperboloid drawn to those sections at the points where their planes touch the sphere being all equal to the radius of that surface, it follows that every pair of rectangular central radii in the central planes parallel to those of each section must be such that the sums of their squared reciprocals will be all equal to nothing, a relation which is true of the whole system of rectangular semidiameters of a conic only in the particular case of the equilateral hyperbola: and again, the cone diverging from the centre and passing through the curve in question, the intersection with the surface of a concentric sphere is, we have seen, always of the second order and always concyclic with the same asymptotic cone, but that cone is the reciprocal of the cone envelope of the above system of central planes. Hence we see that the assemblage of planes which intersect any cone of the second order each in a pair of rectangular sides envelopes always another cone of the second order, of which the reciprocal is always concyclic with the original cone.

These different properties of cones of the second order, as indeed the two general principles by means of which they have been so very indirectly established, are obviously two and two reciprocal to each other, so that, if we pleased, one set of them only need have been investigated, as, by reciprocating them to a sphere of arbitrary radius, the correlatives would have immediately followed. The independent solutions for the two different sets were however, in the present instance, not more difficult for one than for the other, while in most instances they serve to elucidate, and in all cases of doubt to verify each other.

There is another way in which the curve of equal and opposite curvature may be determined on the hyperboloid

of one sheet, which, though the method already given is perhaps as simple as could be desired, is possibly still more simple. Let r be the distance of any point on the curve from the centre of the surface, then, since the tangent plane at the extremity of r must intersect the surface in two right lines at right angles to each other, the parallel section through the centre conjugate to r must be therefore an equilateral hyperbola: of this curve any two conjugate semidiameters taken arbitrarily will obviously form with r a system of three conjugate semidiameters of the surface, therefore the sum of their squares must be equal to the sum of the squares of the three semiaxes; but in every equilateral hyperbola the sum of the squares of every pair of conjugate semidiameters is always equal to nothing, hence r^2 by itself is equal to the sum of the squares of the semiaxes of the surface, and therefore the curve required is the intersection with the hyperboloid of the concentric sphere $r^2 = a^2 + b^2 - c^2$.

On any particular hyperboloid there exists of course but one curve for all the points of which the two generatrices of the surface actually intersect at right angles, but on every hyperboloid of one sheet there exists an infinite number for all the points of which they appear to do so; in fact, to every different point of view assumed arbitrarily in space there corresponds a different curve on the surface—that curve may in all cases be readily determined from the preceding principles, it is always of the fourth order, the intersection with the hyperboloid of a cone of the second order diverging from the point of view.

For, from the assumed point of view, wherever it be, draw two planes passing through the two intersecting generatrices at any point on an hyperboloid of one sheet, or on any other rule surface whatever which admits of a double generation by right lines; if, then, that point be a point on the curve possessing the required property, those two planes must intersect at right angles, but passing each through a generatrix they are both tangent planes to the surface, and therefore to the cone which envelopes it from the point of view as vertex. Hence the curve on any such surface locus of the series of points at which the two intersecting generatrices appear to cross at right angles, is the intersection with the surface of the cone diverging from the point of view locus of the lines of intersection of pairs of rectangular tangent planes to the cone which from the same point as vertex envelopes the surface. In the case of the hyperboloid of

one sheet the enveloping cone is always of the second order, consequently the cone whose intersection with the surface determines the required curve is also always of the second order, and is connected with the enveloping cone from the same vertex by the relation that with that cone its reciprocal is always confocal.

If any plane be drawn arbitrarily intersecting these two cones, the curve in which it intersects the latter is obviously the envelope of the projections from the point of view of all the generatrices of the surface upon that plane, and the curve in which it intersects the other is the locus of the points in which the same projections appear from the same point to intersect two and two at right angles. Hence the envelope of the projections from any point upon any plane of the whole system of rectilinear generatrices of any hyperboloid of one sheet is always a curve of the second order, the intersection with the plane of the cone which from the point as vertex envelopes the surface, or, which is the same thing, the curve contour of the projection of the surface itself: and the locus of the whole system of points on the same plane at which the same projections, viewed from the point, appear to intersect two and two at right angles, is always another curve of the second order connected with the former by means of the relation existing between the two convertical and coaxal cones of which the two curves are sections by the same plane.

Of these two properties the first is indeed evident, and is obviously but a particular case of the very general but equally evident principle, that the projection of any plane section whatever of a surface of the second order, from any point upon any plane has always double contact, real or imaginary, with the curve of the second order bounding the projection of the surface itself from the same point upon the same plane; the points of contact being the projections of the two points, real or imaginary, in which the plane of the section intersects the plane of contact of the cone which from the vertex of projection envelopes the surface.* And this

* More generally still, if from any point any curve whatever traced out upon a surface of any order be projected upon any plane, such curve being the intersection with the surface, suppose of the n^{th} order, with another surface, suppose of the m^{th} order, then always will the projected curve have contact with the contour of the projection of the surface itself at $m.n.(n-1)$ different points, real or imaginary. This is evident, for those points of contact are the projections of the points, real or imaginary, in which the given curve intersects the line of contact of the cone which

being true of every plane section of every surface of the second order, is therefore true of the particular plane sections constituted by all the intersecting pairs of rectilinear generatrices of the hyperboloid of one sheet.

If in any particular case the assumed plane be the plane of contact of the enveloping cone, then obviously will the curve envelope of the whole system of projected generatrices be the curve of contact itself; and if at the same time the point of projection be off at infinity in any direction, then will the enveloping curve be the central section of the surface by the diametral plane conjugate to the direction of the infinitely distant point. This last evidently includes the very particular case noticed by M. Leroy in his geometry of three dimensions, viz. that the curve envelope of the whole system of rectilinear generatrices of the hyperboloid of one sheet projected orthographically upon any one of the three central principal planes of the surface, is always the principal section of the surface in the same principal plane.

In general, if the point of view go off to infinity in any direction, the two cones will of course degenerate into cylinders, but then always one of them will be of a very particular species; the enveloping cone will become a cylinder of some shape or other, depending on the direction of the infinitely distant point of projection and on the particular species of surface enveloped, but the other cone, the locus of the intersections of its system of pairs of rectangular tangent planes, will then obviously become always either a right circular cylinder of revolution, or a plane, according as the principal section of the former perpendicular to its generatrices is a central conic or a parabola; consequently, in the case of the hyperboloid of one sheet it will be always a cylinder of revolution, for the particular class of parabolic enveloping cylinders to surfaces of the second order is confined exclusively to the two paraboloids.

On the hyperboloid of one sheet, therefore, the curve locus of the whole system of points, at which the pairs of intersecting generatrices, viewed from any infinitely distant point, appear to cross two and two at right angles, is always the intersection of the surface with a right circular cylinder of the second order, whose axis of revolution is the diameter passing through the infinitely distant point, and whose base

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from the point of projection as vertex envelopes the given surface, and these latter points are the intersections of three surfaces of the orders m, n, and $n - 1$.

upon any plane perpendicular to the axis is the circle locus of the points of intersection of the whole system of pairs of rectangular tangents to the ellipse or hyperbola contour of the orthographic proportion on the plane of the surface itself, or, which is the same thing, envelope of its whole system of rectilinear generatrices projected orthographically upon the same plane.*

The particular positions of the infinitely distant point, for which the enveloping cylinders are in the transition state between those of the elliptic and of the hyperbolic class, present no exceptions to the general type of this locus on the hyperboloid of one sheet, for the cylinders circumscribing that surface in its general state pass from elliptic to hyperbolic, and *vice versa,* not ever through the particular form of the parabolic cylinder, but always by becoming infinitely flat, when they are themselves either portion indifferently of a plane bounded by two parallel right lines, and when therefore the cylinders loci of the intersections of their pairs of rectangular tangent planes are still all of revolution: this is evident, for that particular system of infinitely distant points consists of those in which the surface is touched by its system of asymptotic tangent planes, and forms the conic in which the asymptotic cone intersects the plane infinity; those particular tangent planes intersect the surface each in a pair of parallel generatrices, one from each of the two opposite systems, and therefore intersected each by all those of the system to which it does not belong; the enveloping cylinders from their points of contact degenerate into the tangent planes themselves, or rather into either indifferently of the two portions of those planes bounded by the two parallel generatrices in which they intersect the surface, and the curves of the second order, outlines of the projections from the same points of the surface itself upon any plane assumed

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* The radius of this cylinder may be readily found in terms of the semiaxes of the surface, and the direction angles of the plane of projection; for, denoting by x and y the semiaxes of the conic contour of the projection of the surface upon the given plane, by p the central perpendicular on the tangent plane parallel to the plane of projection, by r the radius of the cylinder, and by a, b, c, and α, β, γ, the semiaxis and direction angles respectively, we have at once the two relations,

$$r^2 = x^2 + y^2, \text{ and } x^2 + y^2 + p^2 = a^2 + b^2 + c^2,$$

from which we immediately get

$$r^2 = a^2 + b^2 + c^2 - p^2 = a^2 + b^2 + c^2 - a^2 \cos^2 \alpha - b^2 \cos^2 \beta^2 - c^2 \cos \gamma$$
$$= a^2 \sin^2 \alpha + b^2 \sin^2 \beta + c^2 \sin^2 \gamma.$$

arbitrarily in space, become also all infinitely flat, the portions of the right lines in which the same tangent planes intersect the plane of projection bounded by the two points in which those right lines are met by the two parallel generatrices in their respective tangent planes, so that the two opposite systems of generatrices of the surface projected all from any one of that system of points upon any plane will consist of two corresponding systems passing each all through one of the two points in which the two particular generatrices in the tangent plane to the surface at the point of projection pierce the plane of projection, and also the curve on the surface at which, viewed from the same point, they appear to intersect two and two at right angles, will, as in the general case, be still the intersection of the surface with a cylinder of revolution round the diameter passing through the infinitely distant point, for that cylinder is in this case the locus of the lines of intersection of the whole system of planes which pass through the same two parallel generatrices and intersect two and two at right angles to each other.

But in the particular case when the hyperboloid degenerates into an hyperbolic paraboloid, then is the enveloping cylinder, whatever be the directions of its generatrices, always parabolic, and therefore the cylinder locus of the intersection of its system of pairs of rectangular tangent planes degenerates always into a plane; the directrix plane of that surface. Hence, in the case of the hyperbolic paraboloid the curve envelope of its whole system of rectilinear generatrices orthographically projected upon any plane is always a parabola, the contour of the orthographic projection of the surface itself upon the same plane, and the curve on the surface locus of all the points at which the generatrices of the opposite systems, viewed from any infinitely distant point, appear to intersect two and two at right angles, is always a plane conic, the intersection with the surface of the directrix plane of the projecting cylinder.

In the general case when the point of view is taken, not infinitely distant, but anywhere at all, the case of the hyperbolic paraboloid presents no difference from that of the hyperboloid of one sheet, for the enveloping cone, and therefore the cone locus of the intersections of its pairs of rectangular tangent planes possesses in its general state no distinctive peculiarity by which the case of that particular surface might be distinguished from that of the more general class. In general, therefore, the curve envelope of the projections from any point upon any plane of the whole system of generatrices of any hyperbolic paraboloid is a conic in its

general state, the intersection with the plane of the cone which from the point envelopes the surface, and the curve on the surface along which the generatrices of the opposite systems appear to intersect two and two at right angles is in general a curve of the fourth order, the intersection with the surface of a cone of the second order diverging from the point of view, and connected with the enveloping cone from the same vertex by exactly the same relation as in the case of the hyperboloid of one sheet.

There is however one particular position of the assumed point for which the result has nothing analogous in the more general case of the hyperboloid of one sheet, and that is when the point of view is taken at an infinite distance on the axis of the surface, that is, at its infinitely distant centre. We have seen that in the hyperboloid the curve, even when the view was from the centre, was, whether real or imaginary, always determinate; in the paraboloid however it is always in the same case either imaginary or indeterminate, imaginary if the two directive planes of the surface do not intersect at right angles, and indeterminate if they do. This is evident, for the orthographic projections of the two systems of generatrices upon any plane perpendicular to the axis of an hyperbolic paraboloid form always two systems of parallel right lines, parallel to the intersections with the plane of projection of the two directive planes of the surface.

Again, the curve on the hyperboloid of one sheet locus of the system of points at which the two principal curvatures of the surface are equal and opposite, that is, the curve along which its pairs of generatrices actually intersect two and two at right angles, undergoes a corresponding modification from its general type, when the surface passing through that particular form degenerates into an hyperbolic paraboloid, and becomes a plane, not a spherical conic: this is evident, for the concentric sphere, locus of the points of intersection of all the systems of three rectangular tangent planes to the hyperboloid in its general state, the sphere whose intersection with the surface gives the curve of equal and opposite curvature, degenerates by its centre moving off to infinity into a plane perpendicular to the axis of the modified surface; hence, on the hyperbolic paraboloid the curve of equal and opposite curvature is always a plane conic, an hyperbola, in a plane perpendicular to the axis of the surface, similar consequently to its principal section, and having for asymptotes the two right lines in which its plane intersects the two directive planes of the surface.

To find the actual position of this plane, which, being the locus of the assemblage of points of intersection of all the systems of three rectangular tangent planes to the surface, has, from its analogy to the directrix of a parabola, been called the directrix plane of the paraboloid, it is obviously only necessary to find a single point on the surface at which the two generatrices intersect at right angles, for then a plane passing through that point and perpendicular to the axis will be the plane required; hence, if the two directive planes of the surface intersect at right angles, that is, if its two principal parabolic sections are equal in magnitude, the plane in question will be the tangent plane at its vertex, for that plane is obviously in all cases at right angles to the axis, and at that point the two generatrices in that particular case plainly intersect at right angles: but if the two directive planes do not intersect at right angles, then the two principal parabolic sections being unequal in magnitude, taking on the lesser parabola the two points at which the parameter of the curve is equal to the principal parameter of the greater parabola, there will be no difficulty in seeing that at the two points thus determined the two principal curvatures of the surface are always equal and opposite, and therefore a plane passing through those two points and perpendicular to the axis of the surface will be the plane required.*

If in the general principle from which we set out we had not confined our attention to a single surface of the second order, and employed the spheres locus of the assemblage of points of intersection of all its systems of three rectangular tangent planes, and envelope of the assemblage of planes passing through the extremities of all its systems of three rectangular central radii, but if for greater generality we had taken instead any three confocal surfaces, and employed the sphere locus of the assemblage of intersections of all the systems of three rectangular planes, tangents, each to one of the surfaces, and the sphere envelope of the assemblage of planes passing through the extremities of all the systems of three rectangular central radii drawn each to one of their three reciprocal surfaces, we would have extended in the one

* In accordance with this view the position of the plane may be determined very simply by the following construction. On the portion of the axis of the surface intercepted between the two principal foci, take a point distant from either of these foci as far as the vertex of the surface is from the other, the polar plane of that point with respect to the surface will then be the plane required.

case to every system of two confocal cones of the second order, one set of the properties established for a single cone, and in the other case to every system of two concyclic cones of the second order the reciprocal system of properties established for a single cone.

Suppose for instance, in the former case, that one of the three surfaces were an ellipsoid, and the other two hyperboloids of either species; then, taking the curve on the sphere r, along which the common circumscribing developable surface envelopes both it and the ellipsoid, and drawing through the radii of the sphere at all the points of that curve pairs of tangent planes to the two hyperboloids, these systems of four planes must then, from the nature of that locus, be all two and two at right angles to each other, and to the corresponding tangent planes to the sphere, the latter being also all tangent planes to the third confocal surface, but passing all through the centre they must also be all tangent planes to the asymptotic cones of the two hyperboloids; and again, the cone locus of the radii of the sphere would, as before, be in that case also of the second order, and such that its reciprocal would be always confocal with the imaginary asymptotic cone to the ellipsoid, and therefore with the two real asymptotic cones to the two confocal hyperboloids. Hence we see that the cone reciprocal to that formed by the lines of intersection of all the pairs of rectangular planes, tangents to any two confocal cones of the second order, will be always a third cone of the second order confocal with the two original cones.

By using in exactly the same manner the reciprocal property of the sphere $\frac{1}{r}$, we get at once the corresponding property of any two fixed concyclic cones of the second order, reciprocal to that just stated—but it is too obvious to need being even mentioned: both theorems are the most general of their class, and admit of a multitude of particular cases, including among them all that have been given for a single cone; these for the same reason are too evident to require being dwelt on any longer.

The general theorem itself for every pair of confocal cones might also be established in another way, without recourse being had to any particular curve on the sphere r; for, taking arbitrarily any point whatever on that sphere, through that point there may, from the nature of the locus, be drawn an infinite number of systems of three rectangular planes,

tangents, each to one of the three confocal surfaces, whatever be their species, and therefore tangents each to one of the three cones of the second order, which from the same point as vertex envelope the three surfaces. Hence the cone reciprocal to that formed by the intersections of all the pairs of rectangular planes tangents to any two of that system of three cones, is always the third cone of the same system; but every three such cones as from the same vertex enveloping three confocal surfaces of the second order, are always all three coaxal and confocal, and therefore the aforesaid reciprocal for any system of two confocal cones of the second order is always a third cone of the second order confocal with both.

This method, though apparently simpler than the former, is really not nearly as simple, for the property of surfaces of the second order, upon which it has been made to depend, is itself at least as difficult, if not more difficult, to be established than the theorem deduced from it is even in its direct investigation.*

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* In case direct solutions of these two theorems might be more satisfactory to the reader, the author gives the following, which were the first at which he arrived.

1st. For a single cone.

It is a well-known property, very easily proved, of every cone of the second order, that if from any point taken arbitrarily upon either of its focal lines, perpendiculars be let fall upon all its tangent planes, then will the curve locus of the feet of the whole system of perpendiculars be always a circle in a plane perpendicular to the other focal line. (See Professor Graves' *Translation of Professor Chasles' Memoir on Cones of the Second Order.*)

Taking then arbitrarily any point (o) upon either focal line of the given cone, from that point let three perpendiculars, op, oq, or, be let fall, two of them, op, oq, upon any two rectangular tangent planes to the cone, and the third, or, upon their right line of intersection. Those three perpendiculars will obviously lie all in a plane perpendicular to the intersection of the tangent planes, and will form in that plane two adjacent sides, and the conterminous diagonal of a rectangle, and their three extremities p, q, r, will obviously lie all on a sphere having for diameter the intercept of the focal line between the assumed point and the vertex of the cone.

But from the general property just referred to, the two points p and q lie both in a plane perpendicular to the other focal line of the cone, therefore the middle point of the line joining them, that is, the middle point of the perpendicular, or, lies in the same plane, and therefore r itself lies also in another plane parallel to the former and at double its distance from the point o. Hence, for the whole system of pairs of rectangular tangent planes, the locus of the feet of the perpendiculars let fall upon all their lines of intersection from any point taken arbitrarily upon either focal line of the cone is always a circle, the intersection with a sphere of a plane perpendicular to the other focal line.

Hence the whole assemblage of lines of intersection themselves forms always a cone of the second order, whose cyclic planes are always perpen-

In conclusion, the author may observe that though perhaps none of the properties discussed in this paper are new, yet the method by which they have been investigated appears not only to possess the advantage of deducing them all from a single and simple elementary principle familiar to all, but also to furnish solutions for each individual property more simple and rapid than any which he has yet seen anywhere given.

Trinity College, Dublin, April 8, 1847.

GEOMETRICAL DEMONSTRATIONS OF SOME PROPERTIES OF GEODESIC LINES.

By ANDREW S. HART, Fellow of Trinity College, Dublin.

IN the proof of some very interesting properties of the "Lines of Curvature of an Ellipsoid," which appeared in the last number of the *Cambridge and Dublin Mathematical Journal*, Mr. Roberts has referred to a remarkable theorem of M. Gauss, at which I believe that he arrived as a particular result from a more general principle. The following demonstration appears to me to be the most direct and simple of which it is capable.

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dicular to the focal lines of the given cone, and whose reciprocal cone is therefore always confocal with the former.

2nd. For any pair of confocal cones.

A point o, as before, being taken arbitrarily upon either of the two common focal lines, and op, oq, or, being let fall upon any two rectangular planes, tangents, each to one of the cones, and upon their right line of intersection. The only difference between this and the former case will be, that the points p and q, in place of lying both in the same plane, will now lie in two different planes, one for each cone, both, as before, perpendicular to the other focal line, and therefore parallel to each other, for different pairs of rectangular tangent planes; the locus of the middle point of the line joining them will therefore in this case also be a plane parallel to and lying midway between their own planes, and therefore the locus of r will be a fourth parallel plane distant from o by an interval equal to the sum (or the difference, as the case may be) of the distances of the planes of p and q.

But, as before, r lies also on the sphere described on the portion of the focal line as diameter intercepted between the assumed point and the vertex of the cone; hence the locus of that point is always a circle in a plane perpendicular to the other focal line; and hence the assemblage of lines of intersection of the whole system of pairs of rectangular planes forms always a third cone of the second order, whose cyclic planes are always perpendicular to the common focal lines of the two given cones, and whose reciprocal is therefore always a fourth cone of the second order confocal with both.

The theorem, as announced by Mr. Roberts, is evidently equivalent to the following:

"If from any point O on a curved surface two geodesic lines be drawn containing the infinitely small angle $AOB = d\omega$, and if the length $AO = \rho$, and the perpendicular distance $AB = Pd\omega$, and if R, R' be the radii of curvature of the surface at A, then the quantity represented by P is such that $\dfrac{d^2P}{d\rho^2} + \dfrac{P}{RR'} = 0.$"

To prove this theorem let us first suppose that AB is perpendicular to OB as well as to OA; then, since it is the shortest distance between the tangents Aa', Bb' to the geodesic lines, it follows that $\dfrac{dP}{d\rho} = 0$, and to find $\dfrac{d^2P}{d\rho^2}$ it is only necessary to find the value of P corresponding to $\rho + \delta\rho$; for, calling this value $P + \delta P$, we have $\delta P = \frac{1}{2}\dfrac{d^2P}{d\rho^2}\delta\rho^2$, neglecting the higher powers of $\delta\rho$. On the tangents to the geodesic lines take portions $Aa' = Bb' = \delta\rho$, and let $a'b' = P'd\omega$; then, if $d\theta$ be the angle between Bb' and a line parallel to Aa', we have

$$a'b' = \sqrt{(AB^2 + d\theta^2\,\delta\rho^2)} \text{ or } P' - P = \tfrac{1}{2}\frac{d\theta^2}{d\omega^2}\cdot\frac{\delta\rho^2}{P}.$$

But if a normal plane through $a'b'$ cut the geodesic lines in a, b, then aa', bb' will meet at the centre of curvature of the geodesic arc ab, and $ab = (P + \delta P)\,d\omega$; therefore if r be the radius of curvature of this arc, we have, by similar triangles,*

$$\frac{P' - P - \delta P}{P} = \frac{aa'}{r}, \text{ whence } \frac{\delta P}{P} = \tfrac{1}{2}\frac{d\theta^2}{P^2 d\omega^2}\delta\rho^2 - \frac{aa'}{r},$$

Now let r' be the radius of curvature of the geodesic line OA, then $aa' = \dfrac{\delta\rho^2}{2r'}$ (*Euclid*, Book III. Prop. 36), and

$$\frac{\delta P}{P} = \tfrac{1}{2}\delta\rho^2\left(\frac{d\theta^2}{P^2 d\omega^2} - \frac{1}{rr'}\right), \text{ therefore } \frac{d^2P}{d\rho^2} = P\left(\frac{d\theta^2}{P^2 d\omega^2} - \frac{1}{rr'}\right).$$

But if we conceive an infinitely small conic section whose diameters are proportional to the square roots of the radii of curvature of the coincident normal sections, to have its

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* [Viz. $a'b'O'$, abO', where O' is the centre of curvature of the geodesic arc ab.]

centre at A and to touch OB at C, the tangent at C will be parallel to Aa', therefore

$$d\theta = \frac{BC}{r'} \text{ and } \frac{d\theta^2}{P^2 d\omega^2} = \frac{1}{r'^2 \tan^2 ACB};$$

and by the principles of conic sections we have

$$\frac{1}{rr'} - \frac{1}{r'^2 \tan^2 ACB} = \frac{1}{RR'},$$

(in fact if the conic cut AO and AB in D and E, and if the tangent at D meet AB at F, the denominators of these three fractions are proportional to the squares of the triangles DAE, DAF, and of the triangle contained by the semiaxes), whence $\dfrac{d^2P}{d\rho^2} + \dfrac{P}{RR'} = 0.$

Now let OB not be perpendicular to AB, and draw the geodesic BC perpendicular to AB, and on OA take two points a, a' at equal distances $\pm\,\delta\rho$ from A, and draw the arcs abc, $a'c'b'$ parallel to AB, and cutting BO, BC in b, b', c, c', it is evident that $bc - b'c'$ is infinitely small of an order higher than the second, since it would be cypher for the osculating paraboloid whose vertex is at A; therefore, neglecting the third and higher powers of $\delta\rho$, we have

$$ac + a'c' = ab + a'b' = \left(2P + \frac{d^2P}{d\rho^2}\,\delta\rho^2\right) d\omega,$$

whence $$\frac{d^2P}{Pd\rho^2} = \frac{ac + a'c' - 2AB}{AB},$$

which is independent of the direction of OB; therefore in every case

$$\frac{d^2P}{d\rho^2} + \frac{P}{RR'} = 0. \qquad \text{Q. E. D.}$$

Now if O be the umbilic of an ellipsoid, it is easy to deduce from the equation of M. Joachimsthal ($pD = ac$) that each geodesic will, at any point, be touched by the generating line through it of the right cone circumscribing the ellipsoid and passing through that point; but if θ be the angle which the osculating plane at any point makes with the plane of the umbilics, $y\dfrac{d\theta}{\sin\theta}$ is the distance between two consecutive sides of the circumscribing cone, and therefore along the curve of contact $Pd\omega = y\dfrac{d\theta}{\sin\theta}$: but Mr. Roberts has proved that

$y = P\sin\omega$, whence $\dfrac{d\omega}{\sin\omega} = \dfrac{d\theta}{\sin\theta}$, and integrating $\tan\frac{1}{2}\theta = k\tan\frac{1}{2}\omega$, k being constant for a given cone, and in general a function of α the semiangle of the cone.

To find the value of k, we have by differentiation

$$k\frac{dk}{d\alpha} = \frac{d\theta}{\sin\theta d\alpha};$$

and since this is true, whatever be the value of θ, we may suppose $\theta = \frac{1}{2}\pi$, in which case x, y, and z are given as functions of α by the three equations,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad \frac{x^2}{a^4} - \frac{y^2}{b^4}\tan^2\alpha + \frac{z^2}{c^4} = 0,$$

$$\frac{x^2}{a^6} + \frac{y^2}{b^6}\tan^4\alpha + \frac{z^2}{c^6} = \frac{\tan^2\alpha}{a^2c^2},$$

and the differential equation of the geodesic is $a^2z\,dx = c^2x\,dz$, whence the differential coefficient with regard to α, of the distance of this geodesic from the locus of $\theta = \frac{1}{2}\pi$, namely,

$$y\frac{d\theta}{d\alpha} = \frac{a^2z\dfrac{dx}{d\alpha} - c^2x\dfrac{dz}{d\alpha}}{\sqrt{(a^4z^2 + c^4x^2)}},$$

and substituting for $x, y, z, \dfrac{dx}{d\alpha}, \dfrac{dz}{d\alpha}$ their values derived from the above equations,

$$\frac{d\theta}{d\alpha} = \frac{\sqrt{(a^2-b^2)}\sqrt{(b^2-c^2)}}{\sqrt{(a^2\tan^2\alpha + b^2)}\sqrt{(c^2\tan^2\alpha + b^2)}} = k\frac{dk}{d\alpha},$$

and integrating

$$k = \frac{\tan\frac{1}{2}\theta}{\tan\frac{1}{2}\omega} = e^{\sqrt{(a^2-b^2)}\sqrt{(b^2-c^2)}\int_{\frac{1}{2}\pi}^{\alpha}\frac{d\alpha}{\sqrt{(a^2\tan^2\alpha+b^2)}\sqrt{(c^2\tan^2\alpha+b^2)}}}.$$

e being the base of the Napierian logarithms.

It is evident that the integral does not change its sign between the limits $\pm\frac{1}{2}\pi$, its value being zero at the inferior limit; that is to say, in passing from one umbilic to the other; therefore if ω and ω' be the corresponding values of θ, we have $\tan\frac{1}{2}\omega' = m\tan\frac{1}{2}\omega$, m being a constant number different from unity: and since the two factors of the denominator change sign together, in passing through infinity, the integral, for values of $\alpha > -\frac{1}{2}\pi$, always retains the same sign; and when $\alpha = \dfrac{3\pi}{2}$, the geodesic returns to the first

umbilic at an angle ω'' such that $\tan\frac{1}{2}\omega'' = m^2 \tan\frac{1}{2}\omega$, and so it will pass and repass for ever, making a series of angles such that the tangents of their halves are in continued proportion: if the geodesic pass through a vertex of the mean axis, ω and ω' are supplemental, and the geodesic can never pass through both extremities of any diameter which is not in the plane of the umbilics, unless b is equal either to a or c.

It may be observed that if the geodesic did not pass through an umbilic, $d\theta$ would change its sign four times in each revolution, viz. at the points of contact with lines of curvature and at the intersections with the plane of the umbilics, but at the umbilics two of these points coincide.

To prove the theorem of M. Joachimsthal, it is only necessary to observe that if diameters of an ellipsoid be drawn parallel to the tangents of any plane section, and perpendiculars let fall from the centre of the ellipsoid on the tangent planes, the rectangle under each perpendicular and the corresponding diameter is proportional to the sine of the angle between the section and the tangent plane, and is therefore a maximum at the point where the section is normal to the surface; but a geodesic line osculates a series of normal sections, therefore for such a line this rectangle is constant. The same is true of a line of curvature, because that its osculating plane, although not normal to the surface, is inclined at an angle which is a maximum or minimum at the point of contact, according as the direction of the line is that of least or greatest curvature.

Trinity College, Dublin, August 16, 1848.

ON SYMBOLICAL GEOMETRY.

By Sir William Rowan Hamilton.

(*Continued from* p. 225, vol. III.)

On the Tensor of a Geometrical Quotient.

40. The equations (218) (222), of Art. 31, shew that for any two equally long straight lines, a, b, the following relation holds good,

$$\left(\mathrm{S}\,\frac{\mathrm{b}}{\mathrm{a}}\right)^2 - \left(\mathrm{V}\,\frac{\mathrm{b}}{\mathrm{a}}\right)^2 = 1 \,\ldots\ldots\ldots (255);$$

or, more concisely, that

$$\mathrm{T}\,\frac{\mathrm{b}}{\mathrm{a}} = 1 \ldots\ldots\ldots\ldots\ldots\ldots (256),$$

if we introduce a new characteristic T of operation on a geometrical quotient, defined by the general formula,

$$\mathrm{T}\,\frac{\mathrm{b}}{\mathrm{a}} = \sqrt{\left\{\left(\mathrm{S}\,\frac{\mathrm{b}}{\mathrm{a}}\right)^2 - \left(\mathrm{V}\,\frac{\mathrm{b}}{\mathrm{a}}\right)^2\right\}} \ldots\ldots (257);$$

where it is to be observed, that the expression of which the square root is taken is essentially a positive scalar, because the square of *every* scalar is *positive*, while the square of *every* vector is on the contrary a *negative* scalar, by the principles of the 12th article. Hence, generally, for *any two* straight lines a, b, of which the lengths are denoted by $\bar{\mathrm{a}}$, $\bar{\mathrm{b}}$, we have the equation,

$$\mathrm{T}\,\frac{\mathrm{b}}{\mathrm{a}} = \bar{\mathrm{b}} \div \bar{\mathrm{a}} \ldots\ldots\ldots\ldots (258);$$

because the expression (257) is doubled, tripled, or multiplied by any positive number, when the line b is multiplied by the same number, whatever it be, while the line a remains unchanged. This geometrical signification of the expression $\mathrm{T}\frac{\mathrm{b}}{\mathrm{a}}$, may induce us to name that expression the TENSOR of the geometrical quotient $\frac{\mathrm{b}}{\mathrm{a}}$, on which the characteristic T has operated; because this *tensor* is a number which directs us how to *extend* (directly or inversely, that is, in what ratio to lengthen or shorten) the denominator line a, in order to render it *as long* as the numerator line b: and it appears to the writer, that there are other advantages in adopting this name "tensor", with the signification defined by the formula (257). Adopting it, then, we might at once be led to see, by (258), from considerations of compositions of ratios between the lengths of lines, that in any multiplication of geometrical quotients among themselves, "the tensor of the product is equal to the product of the tensors." But to establish this important principle otherwise, we may observe that by the equations (87), (88), (99), (100), of Arts. 11, 13, the vector part γ of the product $c + \gamma$ of any two geometrical quotients, represented by the binomial forms $b + \beta$, $a + \alpha$, is changed to its own opposite, $-\gamma$, while the scalar part c of the same product remains unchanged, when we change the signs of the vector parts β, α, of the two factors, without

changing their scalar parts b, a, and also *invert*, at the same time, the *order* of those factors; in such a manner that either of the two following *conjugate equations* includes the other:

$$\left.\begin{array}{l} c + \gamma = (b + \beta)(a + \alpha) \\ c - \gamma = (a - \alpha)(b - \beta) \end{array}\right\} \ldots\ldots\ldots (259);$$

and these two conjugate equations give, by multiplication,

$$c^2 - \gamma^2 = (b^2 - \beta^2)(a^2 - \alpha^2) \ldots\ldots\ldots (260),$$

because the product $(a + \alpha)(a - \alpha) = a^2 - \alpha^2$ is scalar, so that

$$(a^2 - \alpha^2)(b - \beta) = (b - \beta)(a^2 - \alpha^2).$$

This product, $a^2 - \alpha^2$, of the two *conjugate expressions*, or *conjugate geometrical quotients*, denoted here by

$$a + \alpha, \quad a - \alpha \ldots\ldots\ldots (261),$$

is not only scalar, but is also *positive*; because we have, by the principles of the 12th article, the two inequalities,

$$a^2 > 0, \quad \alpha^2 < 0 \ldots\ldots\ldots (262).$$

Making then, in conformity with (257),

$$\mathrm{T}(a + \alpha) = \mathrm{T}(a - \alpha) = \sqrt{(a^2 - \alpha^2)} \ldots\ldots (263),$$

we see that either of the two conjugate equations (259) gives, by (260),

$$\mathrm{T}(c + \gamma) = \mathrm{T}(b + \beta) . \mathrm{T}(a + \alpha) \ldots\ldots (264);$$

or eliminating $c + \gamma$,

$$\mathrm{T}(b + \beta)(a + \alpha) = \mathrm{T}(b + \beta) . \mathrm{T}(a + \alpha) \ldots\ldots (265).$$

It is easy to extend this result to any number of geometrical quotients, considered as factors in a multiplication; and thus to conclude generally that, as already stated, *the tensor of the product is equal to the product of the tensors*; a theorem which may be concisely expressed by the formula,

$$\mathrm{T}\Pi = \Pi\mathrm{T} \ldots\ldots\ldots (266).$$

On Conjugate Geometrical Quotients.

41. It will be found convenient here to introduce a new characteristic, K, to denote the operation of passing from any geometrical quotient to its *conjugate*, by preserving the scalar part unchanged, but changing the sign of the vector part;

with which new characteristic of operation K, we shall have, generally,

$$\mathrm{K}\,\frac{\mathrm{b}}{\mathrm{a}} = \mathrm{S}\,\frac{\mathrm{b}}{\mathrm{a}} - \mathrm{V}\,\frac{\mathrm{b}}{\mathrm{a}} \ldots\ldots\ldots\ldots (267);$$

or,

$$\mathrm{K}\,(a + \alpha) = a - \alpha \ldots\ldots\ldots\ldots (268),$$

if a be still understood to denote a scalar, but α a vector quotient. The *tensors of two conjugate quotients are equal to each other*, by (263); so that we may write

$$\mathrm{TK}\,\frac{\mathrm{b}}{\mathrm{a}} = \mathrm{T}\,\frac{\mathrm{b}}{\mathrm{a}}, \text{ or briefly, } \mathrm{TK} = \mathrm{T} \ldots\ldots (269);$$

and *the product of any two such conjugate quotients is equal to the square of their common tensor*,

$$\frac{\mathrm{b}}{\mathrm{a}}\,\mathrm{K}\,\frac{\mathrm{b}}{\mathrm{a}} = \left(\mathrm{T}\,\frac{\mathrm{b}}{\mathrm{a}}\right)^2 \ldots\ldots\ldots\ldots (270).$$

By separation of symbols, we may write, instead of (267),

$$\mathrm{K} = \mathrm{S} - \mathrm{V} \ldots\ldots\ldots\ldots\ldots (271),$$

and the characteristic K is a *distributive symbol*, because S and V have been already seen to be such: so that the equations (74) (75), of Art. 10, give now the analogous equations,

$$\mathrm{K}\Sigma = \Sigma\mathrm{K}, \quad \mathrm{K}\Delta = \Delta\mathrm{K} \ldots\ldots\ldots\ldots (272),$$

or in words, *the conjugate of a sum* (of any number of geometrical quotients) *is the sum of the conjugates;* and in like manner, the conjugate of a difference is equal to the difference of the conjugates. But also we have seen, in (178), that

$$1 = \mathrm{S} + \mathrm{V},$$

because a geometrical quotient is always equal to the sum of its own scalar and vector parts; we may therefore now form the following *symbolical expressions for our two old characteristics of operation, in terms of the new characteristic* K,

$$\left.\begin{aligned} \mathrm{S} &= \tfrac{1}{2}(1 + \mathrm{K}) \\ \mathrm{V} &= \tfrac{1}{2}(1 - \mathrm{K}) \end{aligned}\right\} \ldots\ldots\ldots\ldots (273).$$

We may also observe that

$$\mathrm{KK}\,\frac{\mathrm{b}}{\mathrm{a}} = \frac{\mathrm{b}}{\mathrm{a}}, \quad \text{or } \mathrm{K}^2 = 1 \ldots\ldots\ldots (274);$$

the *conjugate of the conjugate* of any geometrical quotient being equal to that quotient itself. Combining (273), (274), we find, by an easy symbolical process, which the formulæ

(272) shew to be a legitimate one,

$$\left.\begin{array}{l} KS = \tfrac{1}{2}(K + K^2) = \tfrac{1}{2}(K + 1) = + S \\ KV = \tfrac{1}{2}(K - K^2) = \tfrac{1}{2}(K - 1) = - V \end{array}\right\} \ldots\ldots (275);$$

and accordingly the operation of *taking the conjugate* has been defined to consist in changing the sign of the vector part, without making any change in the scalar part, of the quotient on which the operation is performed. From (273), (274), we may also infer the symbolical equations,

$$\left.\begin{array}{l} S^2 = \tfrac{1}{4}(1 + K)^2 = \tfrac{1}{2}(1 + K) = S \\ V^2 = \tfrac{1}{4}(1 - K)^2 = \tfrac{1}{2}(1 - K) = V \\ SV = VS = \tfrac{1}{4}(1 - K^2) = 0 \end{array}\right\} \ldots\ldots (276);$$

and in fact, after once separating the scalar and vector parts of any proposed geometrical quotient, no farther separation of the same kind is possible; so that the operation denoted by the characteristic S, if it be again performed, makes no change in the scalar part first found, but reduces the vector part to zero; and, in like manner, the operation V reduces the scalar part to zero, while it leaves unchanged the vector part of the first or proposed quotient. We may note here that the same formulæ give these other symbolical results, which also can easily be verified:

$$KS = SK; \quad KV = VK \ldots\ldots\ldots (277);$$

and $$(S + V)^n = S^n + V^n = S + V = 1 \ldots\ldots (278);$$

at least if the exponent n be any positive whole number, so as to allow a finite and integral development of the symbolic power

$$(S + V)^n = 1^n \ldots\ldots\ldots\ldots (279).$$

With respect to the *geometrical signification* of the relation between conjugate quotients, we may easily see that if c and d denote any two equally long straight lines, and x any scalar coefficient or multiplier, then the two quotients

$$\frac{x\mathrm{c}}{\mathrm{c} + \mathrm{d}}, \quad \frac{x\mathrm{d}}{\mathrm{c} + \mathrm{d}} \ldots\ldots\ldots\ldots (280)$$

will be, in the foregoing sense, *conjugate;* because their *sum* will be a *scalar,* namely x, but their *difference* will be a *vector,* on account of the mutual perpendicularity of the lines c − d and c + d, which are here the diagonals of a rhombus, and of which the latter bisects the angle between the sides c and d of that rhombus. (Compare (209).)

Conversely, if the relation

$$\frac{b'}{a} = K \frac{b}{a} \dots\dots\dots\dots\dots (281),$$

be given, we shall have, by the definition (267) of K,

$$0 = V \frac{b' + b}{a} = S \frac{b' - b}{a} \dots\dots\dots (282);$$

whence it is easy to infer that, *if two conjugate geometrical quotients or fractions be so prepared as to have a common denominator* (a), *their numerators* (b, b′) *will be equally long, and will be equally inclined to the denominator, at opposite sides thereof, but in one common plane with it;* in such a manner that the line a (or − a) *bisects the angle* between the lines b and b′, if these three straight lines be supposed to have all one common origin. We are then conducted, in this way, to a very simple and useful *expression, for* (what may be called) *the reflexion* (b′) *of a straight line* (b), *with respect to another straight line* (a), namely the following:

$$b' = K \frac{b}{a} \times a \dots\dots\dots\dots\dots (283).$$

And whenever we meet with an expression of this form, we shall know that the two lines b and b′ are equally long; and also that if they have a common origin, the angle between them is bisected there by one of the two opposite lines ± a, or by a parallel thereto.

Finally, we may here note that, by the principles of the present article, and of the foregoing one, we have the following expressions, which hold good for any pair of straight lines, a and b:

$$\left.\begin{aligned} \frac{a}{b} &= \left(T \frac{a}{b}\right)^2 K \frac{b}{a} \\ S \frac{a}{b} &= \left(T \frac{a}{b}\right)^2 S \frac{b}{a} \\ V \frac{a}{b} &= -\left(T \frac{a}{b}\right)^2 V \frac{b}{a} \end{aligned}\right\} \dots\dots\dots\dots (284).$$

[*To be continued.*]

NOTES ON HYDRODYNAMICS.

V. ON THE VIS-VIVA OF A LIQUID IN MOTION.

By William Thomson.

1. If a liquid of finite dimensions be set in motion, it will go on moving in a manner determined in general by the circumstances affecting its bounding surface, and the forces which operate through its interior. Now all the forces which are observed in nature to act upon the mass of a liquid at rest, whatever may be the agencies to which it is subjected, are such that if the liquid be enclosed in a fixed envelope they cannot disturb its equilibrium, but are in all cases balanced by the resistance which the fluid pressure experiences from the bounding solid. Hence, if a liquid in motion were acted on by such forces only as it might experience when at rest, its motion within the bounding surface would, by d'Alembert's principle, be entirely independent of the forces operating on its mass. The pressure through the interior and at the bounding surface would, however, in general depend, partly upon these forces, and partly upon the state of motion of the liquid; and therefore, in all cases in which the form of the bounding surface is susceptible of alteration by the pressure of the fluid, the forces through the mass will, by the effect they may thus produce on the form of the bounding surface, exercise an indirect influence on the motion which takes place within it. Thus it is that gravity, which could not affect the motion of a liquid entirely filling a rigid closed vessel, will exercise a most important influence on the motion of a liquid contained in an open vessel, and exposing a free surface to the atmospheric pressure. The same remark is applicable to the forces which Faraday has discovered to be exerted by electrified bodies upon liquid non-conductors, and by magnets or galvanic wires upon ferro-magnetic or diamagnetic liquids; but the internal forces of friction which are found to operate in actual liquids (see Note III., by Mr. Stokes); and the forces experienced by a liquid traversed by electric currents, under the action of a magnet, belong to a different class, and none of the preceding remarks are applicable to them.* In this paper it will

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* The analytical characteristic of this class of forces is, that

$$X dx + Y dy + Z dz,$$

is *not* a complete differential, if $X dm$, $Y dm$, $Z dm$ be the components of the force experienced by an element, dm, of the liquid mass.

generally be understood that the forces considered belong to the former class.

2. The bounding surface of a liquid mass may be given as varying in any arbitrary manner with the time, under the sole condition of containing a constant volume; since, if the liquid be enclosed in a perfectly flexible and extensible envelope, we may clearly, by external agency, mould it arbitrarily at each instant, altering it gradually from one form to another. Hence, among the *data* of a hydrodynamical problem, we may have

$$F(x, y, z, t) = 0 \ldots\ldots\ldots\ldots\ldots (1),$$

as the equation of the bounding surface, where F may denote an arbitrary function of the four variables subject to the single condition that the volume of the surface so represented may be constant, but with besides the practical limitation, that there can be no finite variations of the surface in infinitely small times.* From this we may deduce, as was shewn in Note II., the following equation, which must be satisfied by the components u, v, w, of the fluid velocity at any point (x, y, z) infinitely near the boundary

$$F'(x).u + F'(y).v + F'(z).w + F'(t) = 0 \ldots (a).$$

If we denote by l, m, n, the direction cosines of the normal, and by $H\,dt$ the normal motion of the surface at the point (x, y, z) in the time dt; we shall have (vol. II. p. 91),

$$\left.\begin{aligned} l &= \frac{F'(x)}{R} \\ m &= \frac{F'(y)}{R} \\ n &= \frac{F'(z)}{R} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (2),$$

and

$$H = -\frac{F'(t)}{R} \ldots\ldots\ldots\ldots\ldots (3);$$

so that, when the bounding surface is given at each instant, we may regard these four quantities as known. By introducing them in (a), we may put it under the form,

$$lu + mv + nw = H \ldots\ldots\ldots\ldots (4),$$

which will be convenient in the investigations to follow.

* If it be considered that perfectly impulsive action is practically impossible, the farther limitation ought to be introduced that there can be no finite variations in the "normal velocity" of the surface at any point in infinitely small times.

3. Since, according to the article referred to in § 2, the distance in the neighbourhood of a point (x, y, z), between the surface at the time t and the surface at the time $t+dt$, is $H\,dt$, it follows that at any instant, the form of the bounding surface being known, the value of H may be arbitrarily given at each point of it, subject to the sole condition, that the volume contained within the surface must not be changed during the time dt, a condition expressed by the equation

$$\iint H ds = 0 \ldots\ldots\ldots\ldots\ldots (5),$$

where the integration is to include all the elements, ds, of the surface.

4. The components u, v, w, of the fluid velocity at any point (x, y, z), which, when this point is infinitely near the boundary, must satisfy the preceding equation (4), are, through the whole liquid, subject to the equation

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0 \ldots\ldots\ldots\ldots (6),$$

as was proved in Note I. (vol. ii. p. 286). These two equations, (4) and (5), express all the cinematical relations of the problem.

5. Theorem. *If the bounding surface of a liquid, primitively at rest, be made to vary in a given arbitrary manner, the vis-viva of the entire liquid at each instant will be less than it would be if the liquid had any other motion consistent with the given motion of the bounding surface.*

Let u_1, v_1, w_1, be the components of the fluid velocity at (x, y, z) in any other state of motion cinematically possible at the time dt, which must consequently satisfy the following conditions:

at the surface, $$lu_1 + mv_1 + nw_1 = H \ldots\ldots\ldots (b),$$

through the interior, $$\frac{du_1}{dx} + \frac{dv_1}{dy} + \frac{dw_1}{dz} = 0 \ldots\ldots\ldots\ldots (c):$$

u_1, v_1, w_1, may be taken as any three quantities whatever for which these equations hold.

Let Q be the actual vis-viva of the liquid, and Q_1 the vis-viva it would have if u_1, v_1, w_1 represented its motion. We shall have, calling ρ the density,

$$Q = \rho \iiint (u^2 + v^2 + w^2)\, dx\, dy\, dz \ldots\ldots (7),$$

$$Q_1 = \rho \iiint (u_1^2 + v_1^2 + w_1^2)\, dx\, dy\, dz.$$

From these we deduce, by subtraction, and by an algebraic modification,

$$Q-Q_1=\rho\iiint\{2u(u_1-u)+2v(v_1-v)+2w(w_1-w)+(u_1-u)^2+(v_1-v)^2+(w_1-w)^2\}dx\,dy\,dz.$$

Now, according to the proposition proved by Mr. Stokes in Note v., since the fluid was primitively at rest,

$$u\,dx+v\,dy+w\,dz$$

must be the differential of some function of x, y, z (involving also, in general, t as another independent variable), which we may denote by ϕ; so that we have

$$u=\frac{d\phi}{dx},\quad v=\frac{d\phi}{dy},\quad w=\frac{d\phi}{dz}.$$

Hence $\iiint\{u(u_1-u)+v(v_1-v)+w(w_1-w)\}\,dx\,dy\,dz$

$$=\iiint\left\{\frac{d\phi}{dx}(u_1-u)+\frac{d\phi}{dy}(v_1-v)+\frac{d\phi}{dz}(w_1-w)\right\}dx\,dy\,dz.$$

Integrating the first term by parts with reference to x, the second with reference to y, and the third with reference to z, we reduce the second member to the form

$$\iint\phi.\{(u_1-u)\,dy\,dz+(v_1-v)\,dz\,dx+(w_1-w)\,dx\,dy\}$$

$$-\iiint\phi.\left\{\frac{d(u_1-u)}{dx}+\frac{d(v_1-v)}{dy}+\frac{d(w_1-w)}{dz}\right\}dx\,dy\,dz.$$

The triple integral here vanishes in virtue of equations (b) and (c); and the double integral, which is to be extended over the entire bounding surface, may (see Note I. vol. ii. p. 285) be put under the form

$$\iint\phi.\{(u_1-u)\,l+(v_1-v)\,m+(w_1-w)\,n\}\,ds.$$

This also is equal to nothing in virtue of equations (4) and (b); and hence the definite integral under consideration vanishes. Thus we see that the expression for Q_1-Q becomes reduced to

$$Q_1-Q=\rho\iiint\{(u_1-u)^2+(v_1-v)^2+(w_1-w)^2\}\,dx\,dy\,dz\ldots(8).$$

In this expression the factor of $dx\,dy\,dz$, and consequently the entire integral, is essentially positive, unless $u_1=u$, $v_1=v$, $w_1=w$. Hence, of all the states of motion of the fluid cinematically possible at each instant, any one which differs from the actual state of motion possesses a greater vis-viva. Q. E. D.

6. Cor. 1. *The condition that* u dx + v dy + w dz *must be a complete differential is, in addition to the cinematical relations, sufficient to determine the motion.* For in the preceding

demonstration no other condition was introduced to characterise u, v, w.*

COR. 2. *The motion of the fluid at any time is independent of the preceding motion, and depends solely on the given form and normal motion of the bounding surface at the instant.*

COR. 3. *If the bounding surface, after having been in motion, be brought to rest in any position, the liquid will, at the same instant, be reduced to rest.*

7. The expression for the vis-viva of the liquid may be put into a very remarkable form, by making use of the differential coefficients of ϕ in place of u, v, and w; and then integrating by parts, in the following manner. Thus we have

$$Q = \rho \iiint \left(u \frac{d\phi}{dx} + v \frac{d\phi}{dy} + w \frac{d\phi}{dz} \right) dx\, dy\, dz$$

$$= \rho \iint \phi \,.\, (u\, dy\, dz + v\, dz\, dx + w\, dx\, dy)$$

$$- \rho \iiint \phi \,.\, \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right) dx\, dy\, dz.$$

The triple integral vanishes in virtue of equation (5), and the double integral, extended over the bounding surface, may be modified by the transformation employed in §. 4, so that we have the following expression for the vis-viva,

$$Q = \rho \iint \phi \,.\, H \,.\, ds\dagger \ldots\ldots\ldots (9).$$

The variation of the function ϕ within the bounding surface will not affect the value of this integral, in which ϕ may be considered as merely a function of the coordinates of a point in the surface itself. Hence, while the factor H expresses the given normal velocities at the different points of the containing surface, the other factor, ϕ, under the integral sign, is such as to express by its differential components, with reference to superficial coordinates, the tangential component of the velocity of a particle of the fluid in contact with this surface.

Glasgow College, Jan. 11, 1849.

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* See vol. III. p. 84, where similar reasoning was applied to prove a theorem, of which the corollary in the text is a particular case.

† This is a particular case of a general theorem proved in an article entitled "Propositions in the Theory of Attraction, Part II.," being obtained by taking $R = R_1 = \sqrt{(u^2 + v^2 + w^2)}$ and $\theta = 0$, in equations (3) and (4) of that article (Old Series, vol. III. p. 202).

MATHEMATICAL NOTES.

I.—*On a Solution of a Cubic Equation.*

By James Cockle.

(*Being a Supplement to the Paper at* pp. 28, 29, *of* Vol. III. *Old Series.*)

Let ω represent an unreal cube root of unity, then α, β, and γ are subject to the relation

$$\left(\frac{1}{\alpha}+\frac{\omega}{\beta}+\frac{\omega^2}{\gamma}\right)\left(\frac{1}{\alpha}+\frac{\omega^2}{\beta}+\frac{\omega}{\gamma}\right)=0\ldots\ldots\ldots(a).$$

Again, let x_1, x_2, and x_3 be the roots of the given cubic in x, the coefficients of which we suppose to be real; then we may make

$$\alpha = x_1 - z,\quad \beta = x_2 - z,\quad \gamma = x_3 - z.$$

Suppose that the first factor of (a) is zero, and in it, for α, β, and γ, substitute the values just given; we shall, after the necessary reductions and taking into account the condition

$$1+\omega+\omega^2=0,$$

arrive at the equation

$$(x_1+\omega x_2+\omega^2 x_3)z + x_2x_3 + \omega x_1x_3 + \omega^2 x_1x_2 = 0\ldots.(b).$$

Multiply (b) by $2(x_1+\omega^2 x_2+\omega x_3)$; for ω and ω^2 substitute their values, and for the symmetric functions their equivalents in terms of the coefficients of the given cubic, and we shall find that

$$(6b-2a^2)z - 9c + ab + 3\xi\sqrt{(-1)} = 0,$$

where $\xi = x_1x_2(x_2-x_1)+x_1x_3(x_1-x_3)+x_2x_3(x_3-x_2)$.

Hence, (1) when $\xi=0$, or the given cubic has equal roots, z is real and has only one value; (2) when the roots of the given cubic are real and unequal, ξ is real (and finite), and consequently z unreal; (3) when the given cubic has conjugate unreal roots z is real; for, if we make x_1 and x_2 the unreal roots, we may assume

$$x_1 = p + q\sqrt{(-1)},\quad x_2 = p - q\sqrt{(-1)};$$

and consequently the real part of ξ will be

$$x_3(p^2-q^2-px_3+px_3-p^2+q^2),\ \text{or zero},$$

which shews that ξ is of the form $m\sqrt{(-1)}$, and z is therefore real, since $\sqrt{(-1)}$ will disappear from it. The same results will follow however we interchange x_1, x_2, and x_3, or whichever factor of (b) we equate to zero; (4) when z is real, the equation in x has unreal roots or equal roots; (5) when z is unreal, all the roots of the equation in x are real and unequal.

The method here discussed does not, of course, remove any of the difficulties of the "irreducible case," but I confess that it was in endeavouring to overcome them, that in the early part of the year 1835, I invented this solution.* Let me add that incidental discoveries, arrived at like the present one, and in the pursuit of ends perhaps as hopeless as that which I had in view, may afford an ample reward to those who labour upon such subjects as the solution of equations of the fifth degree—in which success may possibly be unattainable.

2, *Church-Yard Court, Temple, June* 9, 1848.

II.—*Remarks on the Deviation of Falling Bodies to the East and to the South of the Perpendicular; and Corrections of a previous Paper* (vol. III. p. 206) *on the same subject.*

[In the list of *Errata*, at the end of the preceding volume, certain mistakes were indicated as occurring in a paper on the problem of falling bodies. These, besides a much graver error in principle, which could not be satisfactorily indicated in a list of errata, had been pointed out in a note received by the Editor from Mr. Hart; which, having arrived too late for publication in the preceding number, is now laid before the readers of the *Journal*.]

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* The youthful inexperience (and limited acquaintance with the subject) that induced me to enter upon such a course of enquiry, and even for a time to dream of success in it, prevented me from avowing myself as the author, until I had ascertained from competent sources the originality and validity of the solution. I made one or two attempts to attract attention to it at the time, but their want of success, and an absence of a year in the West Indies and America, shortly after caused my attention to be diverted from the subject. I need not say that when I published it, I was fully aware of its *modus operandi* in the irreducible case.

In the last volume of the *Mathematical Journal*, at the close of a very clear and satisfactory account of the easterly* deviation of falling bodies, there is an attempt to calculate the deviation towards the equator, which seems likely to mislead the young mathematicians for whose use it was expressly intended.

The principle on which the calculation rests, is that the falling body moves in the plane of a great circle through the *foot* of the original vertical. This would be true if the vertical were the direction of the earth's attraction; but it is evidently the direction of the resultant of attraction and centrifugal force, and therefore makes an angle with the plane in which the body moves, equal to $\dfrac{c\omega^2}{g}\sin\lambda\cos\lambda$, (the notation of the article referred to being adopted); so that, if h be the height from which the body falls, this plane cuts the earth's surface at a distance $\dfrac{c\omega^2 h}{g}\sin\lambda\cos\lambda$ to the north of the vertical, and it is from this point that the southerly deviation, arising from the inclination of this plane to the

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* [A still simpler investigation of the Easterly deviation is obtained in the following manner, which has the great additional advantage of shewing that the result is applicable to the case of a body falling in a pit, as well as to the case exclusively considered in the former paper. The notation of the paper referred to being followed, we have, by the principle of the uniform description of areas,

$$r^2 d\theta = c^2\cos^2\lambda.\omega.dt.$$

Now, *if we neglect the variation of gravity in the descent of the body*, we have

$$r^2 = (c - \tfrac{1}{2}gt^2)^2$$

$$= c^2 - c.gt^2 \text{ approximately.}$$

Hence,
$$d\theta = \frac{c^2\cos\lambda.\omega}{c^2 - c.gt^2}\,dt = \omega\cos\lambda\,(1 + \frac{g}{c}t^2)\,dt,$$

which gives
$$\theta = \omega\cos\lambda\,(t + \tfrac{1}{3}\frac{g}{c}t^3).$$

Now the easterly deviation is equal to $c\theta - \omega\cos\lambda.t$; and we therefore find for its value, $\frac{1}{3}g\cos\lambda.\omega.t^3$; which agrees with the result of the preceding paper. It is readily seen that the *variation of gravity* referred to in that paper *produces only an effect which is neglected in the final result*, although unnecessary complication is introduced by taking it into account in some steps of the investigation.

To make this investigation completely satisfactory, it ought to be added that a point moving in the same plane with the falling body, and describing with its radius vector an angle $\cos\lambda.\omega.t$ in the time t, although it will actually have a westerly deviation, yet this deviation, being of the same order as the cube of the angle, will be of the same order as other small quantities neglected in the investigation; and therefore $c\theta - c\cos\lambda.\omega.t$ is a correct approximate expression for the easterly deviation of the falling body.—W. T.]

parallel of latitude, should have been reckoned: the approximate value of this southerly deviation is $\frac{c\omega^2 t^2}{2} \sin\lambda \cos\lambda$ (not $\cos^3\lambda$*), and this would be exactly equal to the distance of the plane from the foot of the vertical if the falling body were acted on only by gravity, therefore in this case there would be no deviation towards the equator (as Laplace has already proved); but since the time t is increased by the resistance of the air, $\frac{t^2}{2}$ is greater than $\frac{h}{g}$, and there is a deviation to the south proportional to the difference of these quantities multiplied by the sine of twice the latitude.

I do not know whether this effect of the resistance of the air has ever been accurately calculated, but it is remarkable that Dr. Brinkley, in mentioning the result of the experiment tried at Hamburg, states that the observed deviation to the south was 0.13 inch, and that theory, *not taking into account the resistance of the atmosphere*, gives no perceptible deviation to the south. Now the height was in this case 250 feet, therefore the time of fall, not taking into account the resistance of the atmosphere, would be nearly 4 seconds, and supposing this time to be increased $\frac{1}{20}$ of a second, by the action of the air, the observed deviation to the south would be accounted for, while the deviation to the east would only be increased by about $\frac{1}{230}$ inch. La Place intimates that this experiment was not tried with sufficient accuracy, and it is impossible to verify it by calculation without knowing the form and weight of the falling body; but the observation appears capable of furnishing a very accurate measure of small retardations, and may be on that account worthy of notice.

ANDREW S. HART.

Trinity College, Dublin, October 31, 1848.

[The same opinion with reference to the cause of the observed southerly deviation is expressed in a paper published in the *Annual Report*, 1846, of the Royal Cornwall Polytechnic Society, by Mr. Rundell, the secretary of that institution, where some important experiments on the subject are described, and a clear and correct mechanical explanation is given. As the subject possesses considerable interest, and has already been partially brought before the readers of

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* See Errata at the end of Vol. III.

this Journal, Mr. Rundell's paper, given below, will probably be acceptable.]*

"*On the Deviation of Falling Bodies to the South of the Perpendicular.*
" *By the Secretary.*

"THE remarks of Professor Oersted at the Southampton "Meeting of the British Association on the deflection of falling "bodies to the south, and the variety of opinions entertained "upon this subject by the most eminent men, not only in "regard to its cause, but also as to its real existence, having "attracted my attention, it occurred to me that the deep mines "of Cornwall would afford facilities for repeating experiments "on this subject which had never before been obtained to "the same extent. The height from which the bodies fell "in the experiments of Professor Guglielmani in 1793, was "231 feet (French) only, in those of Dr. Benzenberg, 240 ft., "and in those of Professor Reich, which are accounted the "best, the depth did not exceed 540 feet, while the deep "shafts of some of the Cornish mines would allow a fall "of two and three times the amount of the greatest of these "distances.†

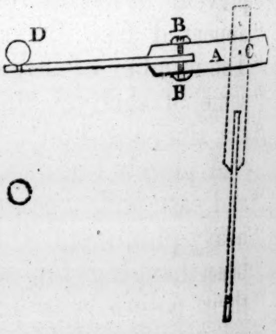
* [Mr. Rundell's paper is given complete in the text; but some notes (not by Mr. Rundell) appended to it are omitted, in which an erroneous view of the entire problem is given, and the same mistake as that committed in the previous paper in this Journal with reference to the southerly deviation is made, although Mr. Rundell himself, in his text, carefully and clearly avoids it.—W. T.]

† Extract from "A letter on the Deviation of Falling Bodies from the Perpendicular, to Sir John Herschel, Bart., from Prof. Oersted," printed in the Report of the British Association for the advancement of Science, for 1846.

"The first experiments of merit upon this subject were made in the last "century, I think in 1793, by Professor Guglielmani. He found in a great "church, an opportunity to make bodies fall from a height of 231 feet. As "the earth rotates from west to east, each point in or upon her describes an "arc proportional to its distance from the axis, and therefore the falling "body has from the beginning of the fall a greater tendency toward east "than the point of the surface, which is perpendicularly below it; thus it "must strike a point, lying somewhat easterly from the perpendicular. "Still the difference is so small, that great heights are necessary for giving "only a deviation of some tenth part of an inch. The experiments of "Guglielmani gave indeed such a deviation; but at the same time they "gave a deviation to the south, which was not in accordance with the "mathematical calculations. De la Place objected to these experiments, "that the author had not immediately verified his perpendicular, but only "some months afterwards. In the beginning of this century, Dr. Benzen-"berg undertook new experiments at Hamburg, from a height of about two "hundred and forty feet. The book in which he describes his experiments "contains, in an appendix, researches and illustrations on the subject, from "Gauss and Olbers, to which several abstracts of older researches are "added. The paper of Gauss is ill printed, and therefore difficult to read; "but the result is that the experiments of Benzenberg should give a devia-

"By the courtesy of the Agents at the United Mines "I was allowed to use their man-engine shaft in some "experiments which were tried in the early part of this "year. It is perpendicular and a quarter of a mile deep, "and this great depth appeared to me, in this case, to be of "some importance, as the deflection was likely to increase "in a greater ratio than the errors arising from imperfection "in the manner of dropping the bodies which are allowed "to fall.

"The value of experiments of this kind depending in a "great measure on the goodness of the method employed "for dropping the bullets or other heavy bodies that may be "used, and not having the means of ascertaining that em-"ployed in the continental experiments, some trouble was "taken to find one that should be as free as possible from "objections, and after many preliminary trials the following "was adopted:—A strong rectangular frame was constructed, "having a shelf or stage inside it, capable of turning freely "upon an axis, supported by pointed centres, fixed in the "sides of the frame. This frame was placed in a horizontal "position over the shaft. The shelf, a section of which is "shewn by *A* in the accompanying "diagram, was held steadily by "buttons, two of which are figured "at *BB*, when these were placed "at right angles across the frame. "The bullets were placed at that "part of the shelf *D* most distant "from the axis *C*, their only sup-"port being the edge of a hole "bored through the shelf at that



"tion of 3.95 French lines. The mean of his experiments gave 3.99; but "they gave a still greater deviation to the south. Though the experiments "here quoted seem to be satisfactory in point of the eastern deviation, I cannot "consider them to be so in truth; for it is but right to state that these "experiments have considerable discrepancies among themselves, and that "their mean therefore cannot be of great value. In some other experiments "made afterwards in a deep pit, Dr. Benzenberg obtained only the eastern "deviation; but they seem not to deserve more confidence. Greater faith "is to be placed in the experiments of Professor Reich, in a pit of 540 feet "at Freiberg. Here the easterly deviation was also found in good agree-"ment with the calculated result; but a considerable southern deviation "was observed. I am not sure that I remember the numbers obtained; "but I must state that they were means of experiments which differed "much among themselves, though not in the same degree as those of "Dr. Benzenberg. Professor Reich has published his researches, an abstract "of which is to be found in Poggendorff's *Annalen der Physik*. After all "this there can be no doubt that our knowledge on this subject is imperfect, "and that new experiments are to be desired."

"place. In the experiments, the bullets having been placed "at *D*, the stage was kept in a horizontal position by the "hand, the lower buttons pressing against the under side "of the frame, and when the moment arrived for dropping "the bullet, its support was suddenly removed by turning "the stage round its axis.

"This plan, it is conceived, ensures the dropping of the "bullet, without an appreciable tendency to any particular "direction arising from the method employed. It may, "perhaps, be objected that the cohesion between the shelf "and the bullet would impart to the latter a motion in the "direction in which the shelf moved. This is the case when "the shelf is made to move very slowly, but when it is "turned suddenly on its axis, even if it be some degrees "from the truly horizontal position, no deviation arises from "this source, as was clearly proved by preceding and sub-"sequent experiments.

"Besides the bullets, iron and steel plummets were used, "the latter being magnetized. In form these were truncated "cones, the lower and larger ends being rounded. These "were suspended by short threads inside a cylinder, to "prevent draughts of air affecting them, and when they "appeared free from oscillation, the threads were let go. "The number of bullets used was forty-eight, and there were "some of each of the following metals, iron, copper, lead, "tin, zinc, antimony, and bismuth.

"A plumb line was suspended at each end of the frame, "and east and west of each other; to these were attached "heavy plummets, the lower ends pointed. After they had "been hanging for some hours in the shaft, a line joining "their points was taken as a datum line from which to "measure the deflection.

"The whole of the bullets and plummets dropped south "of this datum line, and so much to the south that only four "of the bullets fell upon the platform placed to receive them, "the others, with the plummets, falling on the steps of the "man-machine, on the south side of the shaft, in situations "which precluded exact measurements of the distances being "taken. The bullets which fell on the platform were from "10 to 20 inches south of the plumb line.

"The deflection being much greater than I had anticipated "could arise from any cause which appeared likely to produce "a deviation, I feared the whole experiment was a failure, "but more recent considerations have induced me to again "test the method employed: I feel confident that the deflec-

"tion is not due to errors arising from the method of "dropping the bullets, and that it is not at all likely that "draughts of air in the shaft had any important influence on "the result, but that there is a real deflection to the *south* "*of the plumb line*, and that in a fall of one quarter of a mile "it is of no small amount.

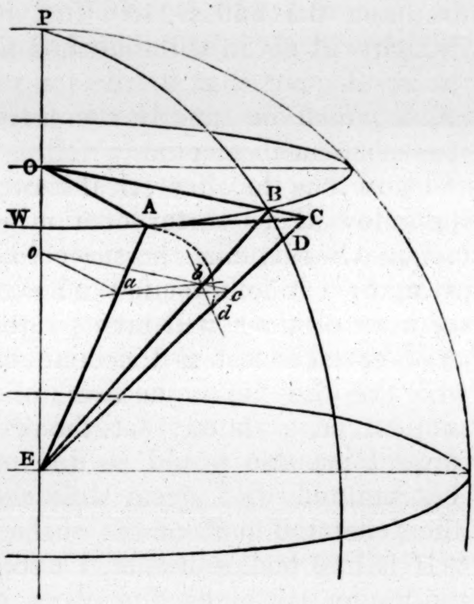
"I now beg to offer for the consideration of the society, "the following, as an explanation of the cause of this pheno-"menon. Seeing that the subject has already been examined "by many eminent men, who seem inclined to doubt the "accuracy of the experiments rather than admit the fact of "an observed southern deflection, and that some among those "who are the best qualified to form an opinion on the "subject, have stated that the deviation arising from the "cause I propose would be inappreciable, this explanation "is submitted with great diffidence, fearing that possibly "some essential part of the subject has been overlooked.

"If falling bodies be acted upon only by the gravitating "and tangential forces, the plane in which any falling body "moves will be indicated by two lines, one line joining the point "from which the body falls and the centre of gravity, and "the other, a line at right angles to this line, forming a "tangent to that part of the circle of latitude which is "situated in the falling body at the *instant it begins to move.* "Taking the earth as a perfect sphere, and the centre of "gravity coinciding with the geometrical centre, this plane "will cut the earth in a great circle, and is of course "stationary, that is, it does not rotate with the earth. Now, "while a falling body is moving forward and downward in "this plane, the point from which it fell is moving round "in the circle of latitude; and the line joining that point and "the centre of gravity lies no longer in this plane, but has "described part of the surface of a cone round the axis of "the earth; consequently the falling body must be some "distance outside this cone, and to the south of the vertical "line passing through the point from which it fell.

"Let A represent a point in the circle of latitude, WAB, "from which a body is about to fall; PE part of the axis "of rotation of the earth, and E its centre: let B be the "position of the point A*, after the lapse of 9^s the time "which, theoretically, a body takes to fall a quarter of a "mile. Through B draw the line OC, in the plane of the

* [More precisely, let B be a point east of the position described in the text by $\frac{1}{3} g \cos l.\omega.t^3$, the amount of the *easterly deviation* of the falling body.—W. T.]

"circle of latitude, make AC a tangent to this circle at A, and "join OA, AE, BE, and "CE; then ABE will "be part of the surface "of the cone spoken of, "and ACE will be part "of the plane, lying "outside the cone, in "which the falling body "will move. Let PD "be a meridian passing "through B, and bd parallel to BD, and a "quarter of a mile deep, "make bc parallel to BC, "and let the dotted line "Ad represent the path "of the falling body; "then bd in the example "here given will not differ to any appreciable extent from "the deflection to the south, of a body falling from A in "the time measured by the angle AOB.



"A quarter of a mile being a small distance compared with "the earth's radius, and the angle BEC being exceedingly "minute, bd may be considered equal to BD, and BD is "equal to BC multiplied by sine of latitude of B, for the "angle BCD equals latitude, and CBD is a right angle. "In this example, the angle AOB is equal to 9s. of time, "and BC the excess of secant over radius of this angle, in "latitude 50 is equal to 2·877 feet, which multiplied into "nat. sine of latitude equals 2·2 feet, the deflection *from the* "*prime vertical*† of a body falling a quarter of a mile in 9s. "in latitude 50.

"As, however, the deflection to be accounted for is one south "of the *plumb line* in north latitude, it is necessary to inquire "how much the plumb line itself is deflected from the line of "gravitation by the centrifugal force. Now, taking the ratio "of the centrifugal force at the equator to be that of gravity "at that place as 1 to 295, which is about a mean between "the values given by the English and French astronomers,

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† [This expression is incorrect, unless a different meaning from that which is usually attached to the term 'vertical' be understood. It would be correct without ambiguity if modified thus: *from a plane passing east and west through the direction of gravitation.*—W. T.]

"we have $\frac{180^\circ \times 60 \times 1}{3{\cdot}14159 \times 295} \cos l \sin l =$ the angle of deflection,
"which will give for the deviation of a plummet hanging by "a line a quarter of a mile long in 50° north lat. a distance "of 2·2 feet, or exactly the same amount as the deflection "to the south of a body falling the same distance in 9^s "But, as 9^s is the theoretical time, or the time a body would "take to fall a quarter of a mile in *vacuo*, and as the air "offers considerable resistance to bodies moving with great "velocities, it follows that a body falling through a quarter "of a mile in air will take longer than 9^s, the angle AOB "will be increased to the same extent, and the falling body "will therefore be to the south of the plumb line. I am not "acquainted with any satisfactory data for calculating the "retardation that would be due to the resistance of the air, "but assuming the whole time the body is falling to be 10^s, "the deflection to the south of the plumb line will be ·5 feet, "if 11^s, 1·09 feet, and if 3^s, 1·6 feet.*

"Besides the deflection of the plumb line from the line "passing through its point of suspension and the centre "of gravity, arising from the rotation of the earth on its axis, "it is further deflected south when considered as normal to "the terrestrial *spheroid*. This deviation is caused by the "attraction of the equatorial protuberance; and as the angle "of the vertical is equal to

$$\frac{180^\circ \times 60 \times 1}{3{\cdot}14159 \times 295} \sin 2l,$$

"or double the angle of deviation due to the earth's rotation, "it follows that these deviations are equal to each other. "Falling bodies will also be deflected south by the same "cause, but it will be seen on examination, that the deflection "arising from this cause will bear no relation to the time "of the fall, but be directly proportional to its distance, con-"sequently no deviation to the south of the plumb line will "be occasioned by it. It seems probable, therefore, that in "experiments on falling bodies in lat. 50° N. with a fall "of a quarter of a mile, the deviation to the south of the "quiescent plumb line will be that which has been given "above, as the amount that would be due, under the given

* [The principle upon which these calculations are founded is that the resistance of the air does not produce any deflection from the primitive plane of motion; and the southerly deviation is determined by finding how much farther the point B is from this plane at the end of the actual time of falling than at the end of 9^s.—W.T.]

"circumstances, to the tangential force acting through a "longer time than the theoretical one, the increased time "being occasioned by the resistance offered by the air.

"I hope to have the pleasure of laying before the Society, "at the next annual exhibition, an account of further experi-"ments on this subject, which will include the times bodies "take falling through given distances, with the deflection south "at the same depths. In these experiments I hope to be "able to avail myself of some very valuable suggestions, "which have been kindly made to me by Sir John Herschel "and Mr. R. W. Fox."*

ON SYMBOLICAL GEOMETRY.

By Sir William Rowan Hamilton.

[*Continued from* p. 89].

Equations of some Geometrical Loci.

42. The equation $$\mathrm{S}\frac{\mathrm{r}}{\mathrm{a}} = 0 \;\ldots\ldots\ldots\ldots\ldots \quad (285),$$

signifies, by what has been already shewn, that the straight line r is perpendicular to a; it is therefore the equation of a *plane* perpendicular to this latter line, and passing through some fixed origin of lines, if r be regarded as a variable line, but a as a fixed line from that origin. The equation

$$\mathrm{S}\frac{\mathrm{r}-\mathrm{a}}{\mathrm{a}} = 0, \text{ or } \mathrm{S}\frac{\mathrm{r}}{\mathrm{a}} = 1 \;\ldots\ldots\ldots \quad (286),$$

[* For supporting and illustrating the views brought forward above, it would be extremely desirable that a series of experiments on various falling bodies, of different dimensions, and on others of different weights but of the same external forms and dimensions, should be made; with the means for ascertaining accurately the times of descent, besides measuring carefully the actual deviations. In the experiments already made by Mr. Rundell the variations in the southerly deviation which his theory would lead us to expect in connexion with the different degrees of retardation by the air, have not been observed except in a few cases which he has mentioned in a letter dated January 13, 1849, in the following terms, in answer to some enquiries I had made: "I may add that a few bullets with small parachutes, *did* fall to the south of the others."

The discrepancies among the results of the various observations described and referred to above have probably arisen in some cases from currents of air, and in others from slight lateral forces ocsasioned by the resistance of the air acting upon bodies not perfectly symmetrical about vertical lines through their centres of gravity, although, from the results described, we may conclude that such accidental disturbing causes are never capable of entirely neutralizing, in carefully performed experiments, the effect of the regular tendency towards the south-east of the plumb line.—W. T.]

expresses, for a similar reason, that the variable line r terminates on another plane, parallel to the former plane, and having the line a for the perpendicular let fall upon it from the origin. If b denote the perpendicular let fall from the same origin upon a third plane, the equation of this third plane will of course be, in like manner,

$$S\frac{r}{b} = 1 \ldots\ldots\ldots\ldots\ldots\ldots (287);$$

and it is not difficult to prove, with the help of the transformations (284), that this other equation,

$$S\frac{r}{b} = S\frac{r}{a} \ldots\ldots\ldots\ldots\ldots (288),$$

represents a fourth plane, which passes through the intersection of the second and third planes just now mentioned, namely, the planes (286), (287), and through the origin.

In general, the equation

$$S\left(\frac{r}{a} + \frac{r}{a'} + \frac{r}{a''} + \&c.\right) = a \ldots\ldots\ldots (289),$$

expresses that r terminates on a fixed plane, if it be drawn from a fixed origin, and if the lines a, a′, a″, &c., and the number *a* be given. It may also be noted here that the equation of the plane which perpendicularly bisects the straight line connecting the extremities of two given lines, a and b, may be thus written:

$$T\frac{r - b}{r - a} = 1 \ldots\ldots\ldots\ldots (290).$$

43. On the other hand, the equation

$$S\frac{r - b}{r - a} = 0 \ldots\ldots\ldots\ldots (291),$$

expresses that the lines from the extremities of a and b to the extremity of r are perpendicular to each other; or that the line r terminates upon a *spheric surface*, in two diametrically opposite points of which surface the lines a and b respectively terminate: and this diameter itself, from the end of a to the end of b, regarded as a *rectilinear locus*, is represented by the equation

$$V\frac{r - b}{r - a} = 0 \ldots\ldots\ldots\ldots (292);$$

which may however be put under other forms. A transformation of the equation (291) is the following:

$$T\left(\frac{2r - b - a}{b - a}\right) = 1 \ldots\ldots\ldots (293);$$

which expresses that the variable radius $r - \frac{1}{2}(b + a)$ has the same length as the fixed radius $\frac{1}{2}(b - a)$. For example, by changing $-a$ to $+b$, in this last equation of the sphere, we find

$$T\frac{r}{b} = 1, \text{ or } \left(S\frac{r}{b}\right) - \left(V\frac{r}{b}\right)^2 = 1 \ldots. (294),$$

as the equation of a spheric surface described about the origin of lines, as centre, with the line b for one of its radii, so as to touch, at the end of this line b, the plane (287). (Comp. (255)).

And a small *circle* of this sphere (294), if it be situated on a secant plane, parallel to this tangent plane (287), which new plane will thus have for its equation,

$$S\frac{r}{b} = x \ldots\ldots\ldots\ldots (295),$$

where x is a scalar, numerically less than unity, and constant for each particular circle, will also be situated on a certain corresponding cylinder of revolution, which will have for its equation

$$\left(V\frac{r}{b}\right)^2 = x^2 - 1 \ldots\ldots\ldots\ldots (296);$$

where $x^2 - 1$ is negative, as it ought to be, by the 12th article, being equal to the square of a vector. The sphere may be regarded as the locus of these small circles; and its equation (294) may be supposed to be obtained by the elimination of the scalar x between the equations of the plane (295), and of the cylinder (296).

44. Conceive now that instead of cutting the cylinder (296) *perpendicularly* in a *circle*, we cut it *obliquely*, in an *ellipse*, by the plane having for equation

$$S\frac{r}{a} = x \ldots\ldots\ldots\ldots (297),$$

where x is the same scalar as before; so that this new plane is parallel to the fixed plane (286), and cuts the plane of the circle (295) in a straight line situated on that other fixed plane (288), which has been seen to contain also the intersection of the same fixed plane (286) with the tangent plane (287). The *locus of the elliptic sections*, obtained from the circular cylinders by this construction, will be an *ellipsoid*; and conversely, an ellipsoid may in general be regarded as such a locus. The equation of the *ellipsoid*, thus found, by eliminating x between the equations (296), (297), is the following:

$$\left(S\frac{r}{a}\right)^2 - \left(V\frac{r}{b}\right)^2 = 1 \ldots\ldots\ldots (298);$$

and by some easy modifications of the process, it may be shewn that a *hyperboloid*, regarded as a certain other locus of ellipses, may in general be represented by an equation of the form

$$\left(\mathrm{S}\,\frac{\mathrm{r}}{\mathrm{a}}\right)^2 + \left(\mathrm{V}\,\frac{\mathrm{r}}{\mathrm{b}}\right)^2 = \mp 1 \;\ldots\ldots\ldots\; (299).$$

The upper sign belongs to a hyperboloid of *one sheet*, but the lower sign to a hyperboloid of *two sheets;* while the *common asymptotic cone* of these two (conjugate) hyperboloids (299) is the locus of a certain other system of ellipses, and is represented by the analogous but intermediate equation,

$$\left(\mathrm{S}\,\frac{\mathrm{r}}{\mathrm{a}}\right)^2 + \left(\mathrm{V}\,\frac{\mathrm{r}}{\mathrm{b}}\right)^2 = 0 \;\ldots\ldots\ldots\; (300).$$

These equations admit of several instructive transformations, to some of which we shall proceed in the following article.

On some Transformations and Constructions of the Equation of the Ellipsoid.

45. The equation (298) of the ellipsoid resolves itself into factors, as follows:

$$\left(\mathrm{S}\,\frac{\mathrm{r}}{\mathrm{a}} + \mathrm{V}\,\frac{\mathrm{r}}{\mathrm{b}}\right)\left(\mathrm{S}\,\frac{\mathrm{r}}{\mathrm{a}} - \mathrm{V}\,\frac{\mathrm{r}}{\mathrm{b}}\right) = 1 \ldots\ldots\; (301);$$

where the sum and difference, which when thus multiplied together give unity for their product, are *conjugate expressions* (in the sense of recent articles); they have therefore a *common tensor*, which must itself be equal to unity; and consequently we may write the equation of the ellipsoid thus,

$$\mathrm{T}\left(\mathrm{S}\,\frac{\mathrm{r}}{\mathrm{a}} + \mathrm{V}\,\frac{\mathrm{r}}{\mathrm{b}}\right) = 1 \;\ldots\ldots\ldots\; (302),$$

where the sign of the vector may be changed. Substituting for the characteristics of operation, S and V, their symbolical values (273), we are led to introduce two new fixed lines g and h, depending on the two former fixed lines a and b, and determined by the equations

$$\frac{\mathrm{r}}{2\mathrm{a}} + \frac{\mathrm{r}}{2\mathrm{b}} = \frac{\mathrm{r}}{\mathrm{g}}; \quad \frac{\mathrm{r}}{2\mathrm{a}} - \frac{\mathrm{r}}{2\mathrm{b}} = \frac{\mathrm{r}}{\mathrm{h}} \ldots\ldots\; (303);$$

and thus the equation of the ellipsoid may be changed from (302) to this other form,

$$\mathrm{T}\left(\frac{\mathrm{r}}{\mathrm{g}} + \mathrm{K}\,\frac{\mathrm{r}}{\mathrm{h}}\right) = 1 \;\ldots\ldots\ldots\; (304);$$

which, by the principles (269), (272), (274), may also be thus written,

$$\mathrm{T}\left(\frac{\mathrm{r}}{\mathrm{h}} + \mathrm{K}\frac{\mathrm{r}}{\mathrm{g}}\right) = 1 \; \ldots\ldots\ldots\ldots \; (305);$$

so that the symbols, g and h, may be interchanged in either of the two last forms of the equation of the ellipsoid.

46. Let $\bar{\mathrm{r}}$, $\bar{\mathrm{g}}$, $\bar{\mathrm{h}}$ be conceived to be numerical symbols, denoting respectively the lengths of the three lines r, g, h; and make, for conciseness,

$$\mathrm{r} \div \bar{\mathrm{r}}^2 = \mathrm{r}'; \quad \mathrm{g} \div \bar{\mathrm{g}}^2 = \mathrm{g}'; \quad \mathrm{h} \div \bar{\mathrm{h}}^2 = \mathrm{h}' \; .\,. \; (306);$$

so that the symbols r′, g′, h′ shall denote three new lines, having the *same directions* as the three former lines r, g, h, but having their *lengths* respectively *reciprocals* of the lengths of those three former lines. Then, by the properties of conjugate quotients already established, we shall have the transformations

$$\frac{\mathrm{r}}{\mathrm{g}} = \mathrm{K}\frac{\mathrm{g}'}{\mathrm{r}'}; \quad \mathrm{K}\frac{\mathrm{r}}{\mathrm{h}} = \frac{\mathrm{h}'}{\mathrm{r}'} \; \ldots\ldots\ldots\ldots \; (307);$$

whereby the equation (304) of the ellipsoid becomes

$$\mathrm{T}\left(\frac{\mathrm{h}'}{\mathrm{r}'} + \mathrm{K}\frac{\mathrm{g}'}{\mathrm{r}'}\right) = 1 \; \ldots\ldots\ldots\ldots \; (308).$$

Let g″ be a new line, not fixed but variable, and determined for each variable direction of r′ or of r by the formula

$$\mathrm{g}'' = \mathrm{K}\frac{\mathrm{g}'}{\mathrm{r}'} \times \mathrm{r}'; \quad \text{or } \mathrm{g}'' = \mathrm{K}\frac{\mathrm{g}'}{\mathrm{r}} \times \mathrm{r} \; \ldots . \; (309);$$

so that this new and variable line g″ is, by what was shewn respecting the expression (283), the *reflexion* of the fixed line g′ with respect to a line having the variable direction just mentioned, of r′ or of r: we may then write the equation (308) of the ellipsoid as follows,

$$\mathrm{T}\,\frac{\mathrm{h}' + \mathrm{g}''}{\mathrm{r}'} = 1 \; \ldots\ldots\ldots\ldots \; (310).$$

And by comparing this with the formula (256), we see that the length of the line r′, or *the reciprocal of the length* $\bar{\mathrm{r}}$ *of the variable semidiameter* r *of the ellipsoid, is equal to the length of the line* h′ + g″; which latter line is the symbolical sum of one fixed line, h′, and of the variable reflexion, g″, of another fixed line, g′; this *reflexion* having been already seen to be performed with respect to the variable radius

vector or semidiameter, r, of the ellipsoid, *of which semidiameter the dependence of the length on the direction admits of being thus represented, or constructed, by a very simple geometrical rule.*

47. To make more clear the conception of this geometrical rule, let A denote the centre of the ellipsoid, which centre is the origin of the variable line r; and let two other fixed points, B and C, be determined by the symbolical equations

$$g' = A - C = AC; \quad h' = B - C = BC \ldots . (311):$$

these two notations, AC and A − C, (of which one has been already used in the *text* of the first article* of this Essay on Symbolical Geometry, while the other was suggested in a *note* to the same early article,) being each designed to denote or signify a straight line drawn *to* the point A from the point C. Let D be a new or fourth point, not fixed but variable, and determined by the analogous equation

$$g'' = C - D = CD \ldots\ldots\ldots\ldots (312):$$

then because, in virtue of the relation (309), the lines g′, g″ are equally long, it follows that the variable point D is situated on the surface of that fixed and *diacentric sphere*, which we may conceive to be described *round* the fixed point C as centre, so as to pass *through* the centre A of the ellipsoid as through a given superficial point of this diacentric sphere. Again, in virtue of the same relation (309), or of the geometrical reflexion which the second formula so marked expresses, the symbolic sum of the two lines, g′, g″, has the direction of the line r, or the exactly contrary direction; in fact, that relation (309) conducts to the following scalar quotient,

$$\frac{g' + g''}{r} = \frac{g'}{r} + K\frac{g'}{r} = 2S\frac{g'}{r} \ldots\ldots (313);$$

and this symbolic sum, g′ + g″, may also, by (311) (312), be thus expressed,

$$g' + g'' = (A - C) + (C - D) = A - D = AD \ldots (314).$$

* It was for the sake of making easier the transition to the notation B − A, which appears to the present writer an expressive one, for the straight line drawn *to* the point B *from* the point A, that he proposed to use, with the *same* geometrical signification, the symbol BA, instead of AB: although it is certainly more usual, and perhaps also more natural, when *direction* is attended to, to employ the latter symbol AB, and not the former BA, to denote the line thus drawn from A to B.

If then we denote by E that variable point on the surface of the ellipsoid at which the line r terminates, so that

$$\mathrm{r} = \mathrm{E} - \mathrm{A} = \mathrm{EA} \ldots\ldots\ldots\ldots (315),$$

we shall have the relation

$$\frac{\mathrm{A} - \mathrm{D}}{\mathrm{E} - \mathrm{A}} = \frac{\mathrm{AD}}{\mathrm{EA}} = 2\mathrm{S}\,\frac{\mathrm{g}'}{\mathrm{r}} = \mathrm{V}^{-1}0 \ldots\ldots\ldots (316),$$

which requires that the three points, A, D, E, should be situated on one common straight line We know then the geometrical position of the auxiliary and variable point D, or have a simple construction for determining this variable point D, as corresponding to any particular point E on the surface of the ellipsoid, when the centre A, and the two other fixed points, B and C, are given; for we see that we have merely to seek the *second intersection* of the semidiameter E − A (or EA) of the ellipsoid, with the surface of the diacentric sphere, the *first* intersection being the centre A itself; since this second point of intersection will be the required point D.

48. But also, by (311) (312), we have

$$\mathrm{h}' + \mathrm{g}'' = (\mathrm{B} - \mathrm{C}) + (\mathrm{C} - \mathrm{D}) = \mathrm{B} - \mathrm{D} = \mathrm{BD} \ldots (317):$$

this line BD has therefore, by (310), the length of the line r'; which length is, by (306), (315) the reciprocal of the length $\bar{\mathrm{r}}$ of the semidiameter EA of the ellipsoid. The lines g, h have generally unequal lengths; and because, by (304) (305), their symbols may be interchanged, we may choose them so that the former shall be the longer of the two, or that the inequality

$$\bar{\mathrm{g}} > \bar{\mathrm{h}} \ldots\ldots\ldots\ldots\ldots\ldots (318)$$

shall be satisfied; and then, by (306), the line g' will, on the contrary, be shorter than the line h', or the fixed point B will be *exterior* to the fixed diacentric sphere. Drawing, then, from this external point B, a tangent to this diacentric sphere, and taking the length of the tangent so drawn for the unit of length, the reciprocal of the length of the line BD, which is considered in (317), will be the length of that other line BD', which has the same direction as BD, but terminates at another variable point D' on the surface of the diacentric sphere; in such a manner that this new variable point D', without generally coinciding with the point D, shall satisfy the two equations,

$$\frac{\mathrm{D}' - \mathrm{B}}{\mathrm{D} - \mathrm{B}} = \mathrm{V}^{-1}0\,; \quad \frac{\mathrm{D}' - \mathrm{C}}{\mathrm{D} - \mathrm{C}} = \mathrm{T}^{-1}1 \ldots\ldots (319);$$

for then the two lines D′ − B, D − B (or D′B, DB) will be, in this or in the opposite order, the whole secant and external part, while the length of the tangent to the sphere has been above assumed as unity. Under these conditions, then, *the lengths of the lines* D′B *and* EA *will be equal,* because they will have the length of the line DB for their common reciprocal; so that we shall have the equation

$$\frac{\mathrm{E} - \mathrm{A}}{\mathrm{D}' - \mathrm{B}} = \mathrm{T}^{-1}1 \ldots\ldots\ldots\ldots (320);$$

or, in a more familiar notation,

$$\overline{\mathrm{AE}} = \overline{\mathrm{BD}'} \ldots\ldots\ldots\ldots\ldots (321).$$

It may be noted here that the new radius D′ − C of the diacentric sphere admits (compare the formula (213)) of being symbolically expressed as follows,

$$\mathrm{D}' - \mathrm{C} = \mathrm{K}\,\frac{g''}{h' + g''} \,.\, (h' + g'') \ldots\ldots (322);$$

and, accordingly, this last expression satisfies the two conditions (319), because it gives

$$\frac{\mathrm{D}' - \mathrm{B}}{\mathrm{D} - \mathrm{B}} = \mathrm{S}\,\frac{h' - g''}{h' + g''} \ldots\ldots\ldots\ldots\ldots (323),$$

and

$$\frac{\mathrm{D}' - \mathrm{C}}{\mathrm{D} - \mathrm{C}} = \mathrm{K}\,\frac{g''}{h' + g''} \,.\, \frac{h' + g''}{-\,g''} \ldots\ldots\ldots (324),$$

of which latter expression the tensor is unity.

49. The remarkably simple formula,

$$\overline{\mathrm{AE}} = \overline{\mathrm{BD}'} \ldots\ldots\ldots\ldots\ldots (321),$$

to which we have thus been conducted for the ellipsoid, admits of being easily translated into the following rule for constructing that important surface; which rule for the *construction of the ellipsoid* does not seem to have been known to mathematicians, until it was communicated by the present writer to the Royal Irish Academy in 1846, as a result of his Calculus of Quaternions, between which and the present Symbolical Geometry a very close affinity exists.

From a fixed point A, *on the surface of a given sphere, draw a variable chord of that sphere,* DA; *let* D′ *be the second point of intersection of the spheric surface with the secant* DB, *which connects the variable extremity* D *of this chord* DA *with a fixed external point* B; *and take the radius vector* EA *equal in length to* D′B, *and in direction either coincident with, or opposite to, the chord* DA: *the locus of the point* E, *thus con-*

structed, will be an ellipsoid, which will have its centre at the fixed point A, *and will pass through the fixed point* B.

The fixed sphere through A, in this construction of the ellipsoid, is the *diacentric sphere* of recent articles; it may also be called a *guide-sphere*, from the manner in which it assists to mark or to *represent the direction*, and at the same time serves to *construct the length* of a variable semidiameter of the ellipsoid; while, for a similar reason, the points D and D′ upon the surface of this sphere may be said to be *conjugate guide-points;* and the chords DA and D′A may receive the appellation of *conjugate guide-chords*. In fact, while either of these two guide-chords of the sphere, for instance (as above) the chord DA, coincides in *direction* with a semidiameter EA of the ellipsoid, the distance $\overline{\mathrm{D'B}}$ of the extremity D′ of the other or conjugate guide-chord, D′A, from the fixed external point B, represents, as we have seen, the *length* of that semidiameter. And that the fixed point B, although exterior to the diacentric sphere, is a superficial point of the ellipsoid, appears from the construction, by conceiving the conjugate guide-point D′ to approach to coincidence with A; for E will then tend to coincide either with the point B itself, or with another point diametrically opposite thereto, upon the surface of the ellipsoid.

50. Some persons may prefer the following mode of stating the same geometrical construction, or the same fundamental property, of the ellipsoid: which other mode also was communicated by the present writer to the Royal Irish Academy in 1846. *If, of a rectilinear quadrilateral* ABED′, *of which one side* AB *is given in length and in position, the two diagonals* AE, BD′ *be equal to each other in length, and intersect* (in D) *on the surface of a given sphere* (with centre C), *of which sphere a chord* AD′ *is a side of the quadrilateral adjacent to the given side* AB, *then the other side* BE, *adjacent to the same given side* AB, *is a chord of a given ellipsoid.*

Thus, denoting still the centre of the sphere by C, while A is still the centre of the ellipsoid, we see that the form, magnitude, and position, of this latter surface are made by the foregoing construction to depend, according to very simple geometrical rules, on the positions of the three points A, B, C; or on the form, magnitude, and position of what may (for this reason) be named the *generating triangle* ABC. Two of the sides of this triangle, namely BC and CA, are perpendicular, as it is not difficult to shew from the construction, to the two *planes of circular section* of the ellipsoid;

and the third side AB is perpendicular to one of the two *planes of circular projection* of the same ellipsoid: this third side AB being the axis of revolution of a circumscribed circular cylinder; which also may be proved, without difficulty, from the construction assigned above. (See Articles 52, 53.) The length $\overline{\text{BC}}$ of the side BC of the triangle, is (by the construction) the semisum of the lengths of the greatest and least semidiameters of the ellipsoid; and the length $\overline{\text{CA}}$ of the side CA is the semidifference of the lengths of those extreme semidiameters, or principal semiaxes, of the same ellipsoid: while (by the same construction) these greatest and least *semiaxes*, or their prolongations, intersect the surface of the diacentric sphere in points which are situated, respectively, on the finite side CB of the triangle ABC itself, and on that side CB prolonged through C. The *mean* semiaxis of the ellipsoid, or the semidiameter perpendicular to the greatest and least semiaxes, is (by the construction) equal in length (as indeed it is otherwise known to be) to the radius of the enveloping cylinder of revolution, or to the radius of either of the two diametral and circular sections: the length of this mean semiaxis is also constructed by the portion $\overline{\text{BG}}$ of the axis of the enveloping cylinder, or of the side BA of the generating triangle, if G be the point, distinct from A, in which this side BA meets the surface of the diacentric sphere. And hence we may derive a simple geometrical signification, or property, of this remaining side BA of the triangle ABC, as respects its length $\overline{\text{BA}}$; namely, that this length is a fourth proportional to the three semiaxes of the ellipsoid, that is to say, to the mean, the least, and the greatest, or to the mean, the greatest, and the least of those three principal and rectangular semiaxes.

On the Law of the Variation of the Difference of the Squares of the Reciprocals of the Semiaxes of a Diametral Section.

51. To give a specimen of the facility with which the foregoing construction serves to establish some important properties of the ellipsoid, we shall here employ it to investigate anew the known and important law, according to which the difference of the squares of the reciprocals of the greatest and least semidiameters, of any plane and diametral section, varies in passing from one such section to another. Conceive then that the ellipsoid itself, and the auxiliary or diacentric sphere which was employed in the

foregoing construction, are both cut by a plane AB′C′, passing through the centre A of the ellipsoid, and having B′ and C′ for the orthogonal projections, upon this secant plane, of the fixed points B and C. The auxiliary or guide-point D comes thus to be regarded as moving on the circumference of a circle, which passes through A, and has its centre at C′: and since the semidiameter EA of the ellipsoid, as being equal in length to D′B, by the formula (321) of Art. 48, (or because these are the two equally long diagonals of the quadrilateral ABED′ of Art. 50), must vary inversely as DB (by an elementary property of the sphere), we are led to seek the difference of the squares of the greatest and least values of DB, or of DB′, since the square of the perpendicular B′B is constant for the section. But the shortest and longest straight lines, D_1B', D_2B', which can thus be drawn to the circumference of the auxiliary circle round C′ (namely the section of the diacentric sphere), from the fixed point B′ in its plane, are those drawn to the extremities D_1, D_2 of that diameter $D_1C'D_2$ which passes through, or tends towards this point B′; in such a manner that the four points $B'D_1C'D_2$ are situated on one straight line. Hence the difference of the squares of D_1B', D_2B', is equal to four times the rectangle under D_1C', or AC′, and B′C′; that is to say, under the projections of the sides AC, BC, of the generating triangle, on the plane of the diametral section. *It is, then, to this rectangle, under these two projections of two fixed lines, on any variable plane through the centre of the ellipsoid, that the difference of the squares of the reciprocals of the extreme semidiameters of the section is proportional.* Hence, in the language of trigonometry, this difference of squares is proportional (as indeed it is well known to be) to the product of the sines of the inclinations of the cutting plane to two fixed planes of circular section; which latter planes are at the same time seen to be perpendicular to the two fixed sides AC, BC, of the generating triangle in the construction.

It seems worth noting here, that the foregoing process proves at the same time this other well-known property of the ellipsoid, that the greatest and least semidiameters of a plane section through the centre are perpendicular to each other; and also gives an easy geometrical rule for *constructing the semiaxes of any proposed diametral section:* for it shews that these semiaxes have the *directions* of the two rectangular guide-chords D_1A, D_2A; while their *lengths* are equal, respectively, to those of the lines $D_1'B$, $D_2'B$.

On the Planes of Circular Section and Circular Projection.

52. It may not be uninstructive to state briefly here some simple geometrical reasonings, by which the line BG of Art. 50 may be shewn to have its length equal to that of the radius of an enveloping cylinder of revolution, as was asserted in that article; and also to the radius of either of the two diametral and circular sections of the ellipsoid. First, then, as to the cylinder: the equation $\overline{AE} = \overline{BD'}$ shews that the rectangle under the two lines AE and BD is constant for the ellipsoid, because the rectangle under BD′ and BD is constant for the sphere; and the point D has been seen to be situated upon the straight line AE (prolonged if necessary). Hence the double area of the triangle ABE, or the rectangle under the fixed line AB, and the perpendicular let fall thereon from the variable point E of the ellipsoid, is always less than the lately mentioned constant rectangle; or than the square of the tangent to the diacentric sphere from B; or, finally, than the rectangle under the same fixed line AB and its constant part GB: except at the limit where the angle ADB is right, at which limit the double area of the triangle ABE becomes equal to the last mentioned rectangle. The ellipsoid is therefore entirely enveloped by that cylinder of revolution which has AB for axis, and $\overline{GB}$ for radius; being situated entirely *within* this cylinder, except for a certain limiting curve or system of points, which are *on* (but not outside) the cylinder, and are determined by the condition that ADB shall be a right angle. This limiting condition determines a *second spherical locus* for the guide-point D, besides the diacentric sphere; it serves therefore to assign a *circular locus* for that point, which circle passes through the centre A of the ellipsoid, because this centre is situated on each of the two spherical loci. And hence by the construction we obtain an *elliptic locus* for the point E, namely the ellipse of contact of the ellipsoid and cylinder; which ellipse presents itself here as the intersection of that enveloping cylinder of revolution with the plane of the circle which has been seen to be the locus of D.—It may also be shewn, geometrically, by pursuing the same construction into its consequences, that the ellipsoid is enveloped by *another* (equal) cylinder of revolution, giving a *second* diametral *plane of circular projection;* the first such plane being (by what precedes) perpendicular to the line AB: and that the axis of this second circular cylinder, or the normal to this second plane of circular projection of the

ellipsoid, is parallel to the straight line which touches, at the centre C of the diacentric sphere, the circle circumscribed about the generating triangle ABC.

53. Again, with respect to the diametral and circular sections of the ellipsoid, considered as results of the construction: if we conceive that the guide-point D, in that construction, approaches in any direction, on the surface of the diacentric sphere, to the centre A of the ellipsoid, the conjugate guide-point D′ must then approach to the point G, because this is the second point of intersection of the side BA of the triangle with the surface of the diacentric sphere, if the point A itself be regarded as the first point of such intersection. Thus, during this approach of D to A, the semidiameter EA of the ellipsoid, having always (by the construction) the direction of $\pm$ DA, and the length of D′B, must tend to touch the diacentric sphere at A, and to have the same fixed length as the line BG, or as the radius of the cylinder. And in this way the construction offers to our notice a *circle* on the ellipsoid, whose radius $= \overline{\mathrm{BG}}$, and whose plane is perpendicular to the side AC of the generating triangle; which side is thus seen to be a *cyclic normal* of the ellipsoid, by this process as well as by that of the 51st article.

Finally, with respect to that *other* cyclic plane which is perpendicular to the side BC of the triangle ABC, it is sufficient to observe that if we conceive the point D′ to revolve in a small circle on the surface of the diacentric sphere, from G to G again, preserving a constant distance from the fixed external point B, then the semidiameter EA of the ellipsoid will retain, by the construction, during this revolution of D′, a constant length $= \overline{\mathrm{BG}}$; while, by the same construction, the guide-chord DA, and the semidiameter EA of the ellipsoid, will at the same time revolve together in a diametral plane perpendicular to BC: in which *second cyclic plane*, therefore, the point E will thus trace out a *second circle* on the ellipsoid, with a radius equal to the radius of the former circle; or to that of the *mean sphere* (constructed on the mean axis as diameter, and containing both the circles hitherto considered); or to the radius of either of the two enveloping cylinders of revolution.—It is evident that if the guide-point D describe any other circle on the diacentric sphere, parallel to this second cyclic plane, the conjugate guide-point D′ will describe another parallel circle, leaving the length $\overline{\mathrm{BD}'} = \overline{\mathrm{EA}}$ unaltered; whence

the known theorem flows at once, that if the ellipsoid be cut by a concentric sphere, the section is a spherical ellipse;* and also that the concentric cyclic cone which rests thereon (being the cone described by the guide-chord DA in the construction) has its two cyclic planes coincident with the two cyclic planes of the ellipsoid.

ON THE TRIPLE TANGENT PLANES OF SURFACES OF THE THIRD ORDER.

By Arthur Cayley.

A surface of the third order contains in general a certain number of straight lines. Any plane through one of these lines intersects the surface in the line and in a conic, *i.e.* in a curve or system of the third order having two double points. Such a plane is therefore a double tangent plane of the surface, the double points or points where the line and conic intersect being the points of contact. By properly determining the plane, the conic will reduce itself to a pair of straight lines. Here the plane intersects the surface in three straight lines, *i.e.* in a curve or system of the third order having three double points, and the plane is therefore a treble tangent plane, the three double points or points of intersection of the lines taken two and two together being the points of contact. The number of lines and treble tangent planes is determined by means of a theorem very easily demonstrated, *viz.* that through each line there may be drawn five (and only five) treble tangent planes. Thus, considering any treble tangent plane, through each of the three lines in this plane there may be drawn (in addition to the plane in question) four treble tangent planes: these twelve new planes give rise to twenty-four

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* This easy mode of deducing, from the author's construction of the ellipsoid, the known spherical ellipses on that surface, was pointed out to him in 1846, by a friend to whom he had communicated that construction, namely by the Rev. J. W. Stubbs, Fellow of Trinity College, Dublin. Several investigations, by the present author, connected with the same construction of the ellipsoid, have appeared in the *Proceedings* of the Royal Irish Academy, (see in particular those for July 1846); and also in various numbers of the (London, Edinburgh, and Dublin) *Philosophical Magazine:* in which magazine several articles on Quaternions have been already published by the writer, and are likely to be hereafter continued, which may on some points be perhaps usefully compared with the present Essay on Symbolical Geometry.

new lines upon the surface, making up with the former three lines, twenty-seven lines upon the surface. It is clear that there can be no lines upon the surface besides these twenty-seven; for since the three lines upon the treble tangent plane are the complete intersection of this plane with the surface, every other line upon the surface must meet the treble tangent plane in a point upon one of the three lines, and must therefore lie in a plane passing through one of these lines, such plane (since it meets the surface in two lines and therefore in a third line) being obviously a treble tangent plane. Hence the whole number of lines upon the surface is twenty-seven; and it immediately follows that the number of treble tangent planes is forty-five. The number of lines upon the surface may also be obtained by the following method, which has the advantage of not assuming *a priori* the existence of a line upon the surface. Imagine the cone having for its vertex a given point not upon the surface and circumscribed about the surface, every double tangent plane of the cone is also a double tangent plane of the surface, and therefore intersects the surface in a straight line (and a conic). And, conversely, if there be any line upon the surface, the plane through this line and the vertex of the cone will be a double tangent plane of the cone. Hence the number of double tangent planes of the cones is precisely that of the lines upon the surface. By the theorems in Mr. Salmon's paper "On the degree of a surface reciprocal to a given one," *Journal*, vol. II. p. 65, the cone is of the sixth order and has no double lines and six cuspidal lines, hence by the formula in Plücker's "Theorie der Algebraischen Curven," p. 211, stated so as to apply to cones instead of plane curves, (viz. n being the order, x the number of double lines, y that of the cuspidal lines, u that of the double tangent planes, then

$$u = \tfrac{1}{2}n(n-2)(n^2-9) - (2x+3y)(n^2-n-6) \\ + 2x(x-1) + 6xy + \tfrac{9}{2}y(y-1).)$$

The number of double tangent planes is twenty-seven, which is therefore also the number of lines upon the surface.

Suppose the equation of one of the treble tangent planes to be $w = 0$, and let $x = 0$, $y = 0$, be the equation of *any* two treble tangent planes intersecting the plane $w = 0$ in two of the lines in which it meets the surface. Let $z = 0$ be the equation of a treble tangent plane meeting $w = 0$ in the remaining line in which it intersects the surface. The equation of the surface of the third order is in every case

of the form $wP + kxyz = 0$, P being a function of the second order, but of the four different planes which the equation $z = 0$ may be supposed to represent, one of them such that the function P resolves itself into the product of a pair of factors, and for the remaining three this resolution into factors does not take place. This will be obvious from the sequel: at present I shall suppose that the plane $z = 0$ is of the latter class, or that $P = 0$ represents a proper surface of the second order. Since $x = 0$, $y = 0$, $z = 0$, are treble tangent planes of the surface, each of these planes must be a tangent plane of the surface of the second order $P = 0$, and this will be the case if we assume

$$P = x^2 + y^2 + z^2 + w^2 + yz\left(mn + \frac{1}{mn}\right) + zx\left(nl + \frac{1}{nl}\right)$$
$$+ xy\left(lm + \frac{1}{lm}\right) + xw\left(l + \frac{1}{l}\right) + yw\left(m + \frac{1}{m}\right) + zw\left(n + \frac{1}{n}\right).$$

And considering x, y, z and w as each of them implicitly containing an arbitrary constant, this is the most general function which satisfies the conditions in question.

We are thus led to the equation of the surface of the third order:

$$U = w\left\{x^2 + y^2 + z^2 + w^2 + yz\left(mn + \frac{1}{mn}\right) + zx\left(nl + \frac{1}{nl}\right) + xy\left(lm + \frac{1}{lm}\right)\right.$$
$$\left. + xw\left(l + \frac{1}{l}\right) + yw\left(m + \frac{1}{m}\right) + zw\left(n + \frac{1}{n}\right)\right\} + kxyz = 0.$$

I have found that by expressing the parameter k in the particular form

$$k = \frac{p^2 - \left(lmn - \dfrac{1}{lmn}\right)^2}{2\left(p - lmn - \dfrac{1}{lmn}\right)},$$

or, as this equation may be more conveniently written,

$$k = \frac{p^2 - \beta^2}{2(p - \alpha)}; \quad \alpha = lmn + \frac{1}{lmn}, \quad \beta = lmn - \frac{1}{lmn}.^*$$

* A somewhat more elegant form is obtained by writing $p = 2q + \alpha$; this gives

$$k = \frac{2}{q} \cdot (q + lmn)\left(q + \frac{1}{lmn}\right), \text{ \&c.}$$

The equations of all the planes are expressible in a rational form. These equations are in fact the following:

(w) $w = 0$.

(θ) $lx + my + nz + w\left[1 + \frac{1}{k}\left(l - \frac{1}{l}\right)\left(m - \frac{1}{m}\right)\left(n - \frac{1}{n}\right)\right] = 0,$

($\bar{\theta}$) $\frac{x}{l} + \frac{y}{m} + \frac{z}{n} \quad + w\left[1 - \frac{1}{k}\left(l - \frac{1}{l}\right)\left(m - \frac{1}{m}\right)\left(n - \frac{1}{n}\right)\right] = 0.$

(x) $x = 0,$

(y) $y = 0,$

(z) $z = 0.$

(ξ) $x + \frac{1}{k}\left(m - \frac{1}{m}\right)\left(n - \frac{1}{n}\right) w = 0,$

(η) $y + \frac{1}{k}\left(n - \frac{1}{n}\right)\left(l - \frac{1}{l}\right) w = 0.$

(ζ) $z + \frac{1}{k}\left(l - \frac{1}{l}\right)\left(m - \frac{1}{m}\right) w = 0,$

($\mathfrak{f}$) $lx + \frac{y}{m} + \frac{z}{n} + w = 0,$

($\mathfrak{g}$) $\frac{x}{l} + my + \frac{z}{n} + w = 0,$

($\mathfrak{h}$) $\frac{x}{l} + \frac{y}{m} + nz + w = 0.$

($\bar{\mathfrak{f}}$) $\frac{x}{l} + my + nz + w = 0,$

($\bar{\mathfrak{g}}$) $lx + \frac{y}{m} + nz + w = 0,$

($\bar{\mathfrak{h}}$) $lx + my + \frac{z}{n} + w = 0.$

($\mathfrak{x}$) $x + \frac{l.(p - \alpha) + 2mn}{p + \beta} w = 0,$

($\mathfrak{y}$) $y + \frac{m.(p - \alpha) + 2nl}{p + \beta} w = 0,$

($\mathfrak{z}$) $z + \frac{n.(p - \alpha) + 2lm}{p + \beta} w = 0.$

$$(\bar{\mathrm{x}}) \quad x + \frac{\frac{1}{l}(p-a) + \frac{2}{mn}}{p-\beta} w = 0,$$

$$(\bar{\mathrm{y}}) \quad y + \frac{\frac{1}{m}(p-a) + \frac{2}{nl}}{p-\beta} w = 0,$$

$$(\bar{\mathrm{z}}) \quad z + \frac{\frac{1}{n}(p-a) + \frac{2}{lm}}{p-\beta} w = 0.$$

$$(\mathrm{l}) \quad -\frac{2n}{m(p-a)} x + \frac{1}{m} y + nz + w = 0,$$

$$(\mathrm{m}) \quad lx - \frac{2l}{n(p-a)} y + \frac{1}{n} z + w = 0,$$

$$(\mathrm{n}) \quad \frac{1}{l} x + my - \frac{2m}{l(p-a)} z + w = 0.$$

$$(\bar{\mathrm{l}}) \quad -\frac{2m}{n(p-a)} x + my + \frac{1}{n} z + w = 0,$$

$$(\bar{\mathrm{m}}) \quad \frac{1}{l} x - \frac{2n}{l(p-a)} y + nz + w = 0,$$

$$(\bar{\mathrm{n}}) \quad lx + \frac{1}{m} y - \frac{2l}{m(p-a)} z + w = 0.$$

$$(\mathrm{l}_{,}) \quad -\frac{n(p-a)}{2m} x + \frac{y}{m} + nz + w = 0,$$

$$(\mathrm{m}_{,}) \quad lx - \frac{l(p-a)}{2n} y + \frac{1}{n} z + w = 0,$$

$$(\mathrm{n}_{,}) \quad \frac{1}{l} x + my - \frac{m(p-a)}{2l} z + w = 0.$$

$$(\bar{\mathrm{l}}_{,}) \quad -\frac{m(p-a)}{2n} x + my + \frac{1}{n} z + w = 0,$$

$$(\bar{\mathrm{m}}_{,}) \quad \frac{1}{l} x - \frac{n(p-a)}{2l} y + nz + w = 0,$$

$$(\bar{\mathrm{n}}_{,}) \quad lx + \frac{1}{m} y - \frac{l(p-a)}{2m} z + w = 0.$$

$$(\mathrm{p})\quad -\frac{2x}{p-a}+ny+mz+\left[mn(p-a)-2l(1-m^2-n^2)\right]\frac{w}{p+\beta}=0,$$

$$(\mathrm{q})\quad nx-\frac{2y}{p-a}+lz+\left[nl(p-a)-2m(1-n^2-l^2)\right]\frac{w}{p+\beta}=0,$$

$$(\mathrm{r})\quad mx+ly-\frac{2z}{p-a}+\left[lm(p-a)-2n(1-l^2-m^2)\right]\frac{w}{p+\beta}=0.$$

$$(\bar{\mathrm{p}})\quad -\frac{2x}{p-a}+\frac{1}{n}y+\frac{1}{m}z+\left[\frac{1}{mn}(p-a)-\frac{2}{l}\left(1-\frac{1}{m^2}-\frac{1}{n^2}\right)\right]\frac{w}{p-\beta}=0,$$

$$(\bar{\mathrm{q}})\quad \frac{1}{n}x-\frac{2y}{p-a}+\frac{1}{l}z+\left[\frac{1}{nl}(p-a)-\frac{2}{m}\left(1-\frac{1}{n^2}-\frac{1}{l^2}\right)\right]\frac{w}{p-\beta}=0,$$

$$(\bar{\mathrm{r}})\quad \frac{1}{m}x+\frac{1}{l}y-\frac{2z}{p-a}+\left[\frac{1}{lm}(p-a)-\frac{2}{n}\left(1-\frac{1}{l^2}-\frac{1}{m^2}\right)\right]\frac{w}{p-\beta}=0.$$

$$(\mathrm{p}_,)\quad -\frac{p-a}{2}x+\frac{y}{n}+\frac{z}{m}-lmn\left[\frac{1}{l}\left(1-\frac{1}{m^2}-\frac{1}{n^2}\right)(p-a)-\frac{2}{mn}\right]\frac{w}{p+\beta}=0,$$

$$(\mathrm{q}_,)\quad \frac{x}{n}-\frac{p-a}{2}y+\frac{z}{l}-lmn\left[\frac{1}{m}\left(1-\frac{1}{n^2}-\frac{1}{l^2}\right)(p-a)-\frac{2}{nl}\right]\frac{w}{p+\beta}=0,$$

$$(\mathrm{r}_,)\quad \frac{x}{m}+\frac{y}{l}-\frac{p-a}{2}z-lmn\left[\frac{1}{n}\left(1-\frac{1}{l^2}-\frac{1}{m^2}\right)(p-a)-\frac{2}{lm}\right]\frac{w}{p+\beta}=0.$$

$$(\bar{\mathrm{p}}_,)\quad -\frac{p-a}{2}x+ny+mz-\frac{1}{lmn}\left[l(1-m^2-n^2)(p-a)-2mn\right]\frac{w}{p-\beta}=0,$$

$$(\bar{\mathrm{q}}_,)\quad nx-\frac{p-a}{2}y+lz-\frac{1}{lmn}\left[m(1-n^2-l^2)(p-a)-2nl\right]\frac{w}{p-\beta}=0,$$

$$(\bar{\mathrm{r}}_,)\quad mx+ly-\frac{p-a}{2}z-\frac{1}{lmn}\left[n(1-l^2-m^2)(p-a)-2lm\right]\frac{w}{p-\beta}=0.$$

In fact, representing the several functions on the left-hand side of these equations respectively by the letters placed opposite to them respectively, the function U is expressible in the sixteen forms following:

$$\begin{aligned} U &= w\mathrm{f}\bar{\mathrm{f}}+k\xi yz,\\ &= w\mathrm{g}\bar{\mathrm{g}}+k\eta zx,\\ &= w\mathrm{h}\bar{\mathrm{h}}+k\zeta xy,\\ &= w\theta\bar{\theta}+k\xi\eta\zeta,\end{aligned}$$

$$
\begin{aligned}
U &= w\,\mathrm{l}\,\bar{\mathrm{l}}_{,} + k\mathrm{y}\,\bar{\mathrm{z}}\,x,\\
&= w\mathrm{m}\bar{\mathrm{m}}_{,} + k\bar{\mathrm{z}}\mathrm{x}y,\\
&= w\mathrm{n}\bar{\mathrm{n}}_{,} + k\mathrm{x}\bar{\mathrm{y}}z,\\
&= w\mathrm{l}_{,}\bar{\mathrm{l}} + k\bar{\mathrm{y}}\mathrm{z}x,\\
&= w\mathrm{m}_{,}\bar{\mathrm{m}} + k\bar{\mathrm{z}}\mathrm{x}y,\\
&= w\mathrm{n}_{,}\bar{\mathrm{n}} + k\bar{\mathrm{x}}\mathrm{y}z,\\
&= w\mathrm{p}\mathrm{p}_{,} + k\xi\mathrm{y}\mathrm{z},\\
&= w\mathrm{q}\mathrm{q}_{,} + k\eta\mathrm{z}\mathrm{x},\\
&= w\mathrm{r}\mathrm{r}_{,} + k\zeta\mathrm{x}\mathrm{y},\\
&= w\bar{\mathrm{p}}\,\bar{\mathrm{p}}_{,} + k\xi\bar{\mathrm{y}}\,\bar{\mathrm{z}},\\
&= w\mathrm{q}\bar{\mathrm{q}}_{,} + k\eta\,\bar{\mathrm{z}}\bar{\mathrm{x}},\\
&= w\mathrm{r}\bar{\mathrm{r}}_{,} + k\zeta\bar{\mathrm{x}}\bar{\mathrm{y}},
\end{aligned}
$$

(being the forms containing w, out of a complete system of one hundred and twenty different forms).

The forty-five planes pass five and five through the twenty-seven lines in the following manner:

$(a_1).(w, x, \xi, \mathrm{x}, \bar{\mathrm{x}})$ $(a_4).(x, \mathrm{g}, \bar{\mathrm{h}}, \bar{\mathrm{l}}, \bar{\mathrm{l}}_{,})$ $(a_7).(\mathrm{x}, \mathrm{m}_{,}, \mathrm{n}, \mathrm{q}_{,}, \mathrm{r})$

$(b_1).(w, y, \eta, \mathrm{y}, \bar{\mathrm{y}})$ $(b_4).(y, \mathrm{h}, \bar{\mathrm{f}}, \bar{\mathrm{m}}, \bar{\mathrm{m}}_{,})$ $(b_7).(\mathrm{y}, \mathrm{n}_{,}, \mathrm{l}, \mathrm{r}_{,}, \mathrm{p})$

$(c_1).(w, z, \zeta, \mathrm{z}, \bar{\mathrm{z}})$ $(c_4).(z, \mathrm{f}, \bar{\mathrm{g}}, \bar{\mathrm{n}}, \bar{\mathrm{n}}_{,})$ $(c_7).(\mathrm{z}, \mathrm{l}_{,}, \mathrm{m}, \mathrm{p}_{,}, \mathrm{q})$

$(a_2).(\xi, \bar{\mathrm{f}}, \theta, \bar{\mathrm{p}}, \mathrm{p}_{,})$ $(a_5).(x, \bar{\mathrm{g}}, \mathrm{h}, \mathrm{l}, \mathrm{l}_{,})$ $(a_8).(\bar{\mathrm{x}}, \bar{\mathrm{m}}_{,}, \bar{\mathrm{n}}, \bar{\mathrm{q}}_{,}, \bar{\mathrm{r}})$

$(b_2).(\eta, \bar{\mathrm{g}}, \theta, \bar{\mathrm{q}}, \mathrm{q}_{,})$ $(b_5).(y, \bar{\mathrm{h}}, \mathrm{f}, \mathrm{m}, \mathrm{m}_{,})$ $(b_8).(\bar{\mathrm{y}}, \bar{\mathrm{n}}_{,}, \bar{\mathrm{l}}, \bar{\mathrm{r}}_{,}, \bar{\mathrm{p}})$

$(c_2).(\zeta, \bar{\mathrm{h}}, \theta, \bar{\mathrm{r}}, \mathrm{r}_{,})$ $(c_5).(z, \bar{\mathrm{f}}, \mathrm{g}, \mathrm{n}, \mathrm{n}_{,})$ $(c_8).(\bar{\mathrm{z}}, \bar{\mathrm{l}}_{,}, \bar{\mathrm{m}}, \bar{\mathrm{p}}_{,}, \bar{\mathrm{q}})$

$(a_3).(\xi, \mathrm{f}, \bar{\theta}, \mathrm{p}, \bar{\mathrm{p}}_{,})$ $(a_6).(\mathrm{x}, \mathrm{m}, \mathrm{n}_{,}, \mathrm{q}, \mathrm{r}_{,})$ $(a_9).(\bar{\mathrm{x}}, \mathrm{m}, \mathrm{n}_{,}, \bar{\mathrm{q}}, \bar{\mathrm{r}}_{,})$

$(b_3).(\eta, \mathrm{g}, \bar{\theta}, \mathrm{q}, \bar{\mathrm{q}}_{,})$ $(b_6).(\mathrm{y}, \mathrm{n}, \mathrm{l}_{,}, \mathrm{r}, \mathrm{p}_{,})$ $(b_9).(\bar{\mathrm{y}}, \mathrm{n}, \mathrm{l}_{,}, \bar{\mathrm{r}}, \bar{\mathrm{p}}_{,})$

$(c_3).(\zeta, \mathrm{h}, \bar{\theta}, \mathrm{r}, \bar{\mathrm{r}}_{,})$ $(c_6).(\mathrm{z}, \mathrm{l}, \mathrm{m}_{,}, \mathrm{p}, \mathrm{q}_{,})$ $(c_9).(\bar{\mathrm{z}}, \mathrm{l}, \mathrm{m}_{,}, \bar{\mathrm{p}}, \bar{\mathrm{q}}_{,})$

where each line may be represented by the letter placed oppositive to the system of planes passing through it. The twenty-seven lines lie three and three upon the forty-five planes in the following manner:

(w)	$a_1b_1c_1$	(f)	$a_3b_5c_4$	(l)	$a_5b_7c_9$	(p)	$a_3b_7c_6$
(θ)	$a_2b_2c_2$	(g)	$b_3c_5a_4$	(m)	$b_5c_7a_9$	(q)	$b_3c_7a_6$
$(\bar{\theta})$	$a_3b_3c_3$	(h)	$c_3a_5b_4$	(n)	$c_5a_7b_9$	(r)	$c_3a_7b_6$
(x)	$a_1a_4a_5$	$(\bar{\mathrm{f}})$	$a_2b_4c_5$	$(\bar{\mathrm{l}})$	$a_4b_8c_6$	$(\bar{\mathrm{p}})$	$a_2b_8c_9$
(y)	$b_1b_4b_5$	$(\bar{\mathrm{g}})$	$b_2c_4a_5$	$(\bar{\mathrm{m}})$	$b_4c_8a_6$	$(\bar{\mathrm{q}})$	$b_2c_8a_9$
(z)	$c_1c_4c_5$	$(\bar{\mathrm{h}})$	$c_2a_4b_5$	$(\bar{\mathrm{n}})$	$c_4a_8b_6$	$(\bar{\mathrm{r}})$	$c_2a_8b_9$
(ξ)	$a_1a_2a_3$	(x)	$a_1a_6a_7$	$(\mathrm{l}_,)$	$a_5b_9c_7$	$(\mathrm{p}_,)$	$a_2b_6c_7$
(η)	$b_1b_2b_3$	(y)	$b_1b_6b_7$	$(\mathrm{m}_,)$	$b_5c_9a_7$	$(\bar{\mathrm{q}}_,)$	$b_2c_6a_7$
(ζ)	$c_1c_2c_3$	(z)	$c_1c_6c_7$	$(\mathrm{n}_,)$	$c_7a_9b_7$	$(\mathrm{r}_,)$	$c_2a_6b_7$
		$(\bar{\mathrm{x}})$	$a_1a_8a_9$	$(\bar{\mathrm{l}}_,)$	$a_4b_6c_8$	$(\bar{\mathrm{p}}_,)$	$a_3b_9c_8$
		$(\bar{\mathrm{y}})$	$b_1b_8b_9$	$(\bar{\mathrm{m}}_,)$	$b_4c_6a_8$	$(\bar{\mathrm{q}}_,)$	$b_3c_9a_8$
		$(\bar{\mathrm{z}})$	$c_1c_8c_9$	$(\bar{\mathrm{n}}_,)$	$c_4a_6b_8$	$(\bar{\mathrm{r}}_,)$	$c_3a_9b_8$.

The preceding method was the one that first occurred to me, and which appears to conduct most simply to the actual analytical expressions for the forty-five planes; but it is worth noticing that the relations between the lines and planes might have been obtained almost without algebraical developments, if we had supposed that P, instead of representing a proper surface of the second order, had represented a pair of planes. This would have conducted at once to one of the one hundred and twenty forms U *e.g.* $U = w\theta\bar{\theta} + k\xi\eta\zeta$. Or changing the notation so as to include k in one of the linear functions $U = ace - bdf$, and it is indeed obvious *à priori*, by merely reckoning the number of arbitrary constants, that any function of the third order can be put under this form. If we suppose $a = \mu b$ to be the equation of one of the treble tangent planes through the intersection of the planes a and b, the plane $a = \mu b$ meets the surface in the same lines in which it meets the hyperboloid $\mu ce - df = 0$, *i.e.* the two lines in the plane are generating lines of different species, and consequently one of them meets the pair of lines cd and ef, and the other of them meets the pair of lines cf and de (where cd represents the line of intersection of the planes $c = 0$, $d = 0$, &c.) This suggests a notation for the lines in question, viz. each line may be represented by the three lines which it meets, or by the symbols $ab.cd.ef$ and $ab.cf.de$. Or observing that μ has three values, and

that the same considerations apply mutatis mutandis to the planes through bc and ca, the whole system of lines may be represented by the notation,

$$\begin{array}{ccc} ab, & ad, & af \\ cb, & cd, & cf \\ eb, & ed, & ef \end{array}$$

$$\begin{array}{ccc} (ab.cd.ef)_1, & (ab.cd.ef)_2, & (ab.cd.ef)_3 \\ (ad.cf.eb)_1, & (ad.cf.eb)_2, & (ad.cf.eb)_3 \\ (af.cb.ed)_1, & (af.cb.ed)_2, & (af.cb.ed)_3 \end{array}$$

$$\begin{array}{ccc} (ab.cf.ed)_1, & (ab.cf.ed)_2, & (ab.cf.ed)_3 \\ (ad.cb.ef)_1, & (ad.cb.ef)_2, & (ad.cb.ef)_3 \\ (af.cd.eb)_1, & (af.cd.eb)_2, & (af.cd.eb)_3, \end{array}$$

where the last eighteen lines have been divided into two systems of nine each. The five planes through $(ab.cd.ef)_1$ may be considered as cutting the surface in

$$\begin{array}{c} ab.(ab.cf.ed)_1 \\ cd.(af.cd.eb)_1 \\ ef.(ad.cb.ef)_1 \\ (ad.cf.eb)_2, \quad (af.cb.ed)_3 \\ (ad.cf.eb)_3, \quad (af.cb.ed)_2, \end{array}$$

(which supposes however that the distinguishing suffixes 1, 2, 3, are added to the different planes according to a certain rule). And similarly for the lines in which the planes through the other lines represented by symbols of the like form. The five planes through ab intersect the surface in the lines

$$\begin{array}{cc} c, & e, \\ d, & f, \\ (ab.cd.ef)_1, & (ab.cf.ed)_1 \\ (ab.cd.ef)_2, & (ab.cf.ed)_2 \\ (ab.cd.ef)_3, & (ab.cf.ed)_3. \end{array}$$

And similarly for the planes through the other lines represented by symbols of a like form. Observing that ξ, η, ζ, correspond to b, d, f, and w, θ, $\bar{\theta}$, to a, c, e respectively, ab corresponds to the intersection of w and ξ, *i.e.* to a_1, &c.; also $(ab.cd.ef)_1$, $(ab.cd.ef)_2$, $(ab.cd.ef)_3$ correspond to three lines meeting a_1, b_2, and c_3, *i.e.* to a_5, a_7, a_9, &c.; and the system of the twenty-seven lines as last written down corresponds to the system,

$$\begin{array}{lll} a_1, & b_1, & c_1, \\ a_2, & b_2, & c_2, \\ a_3, & b_3, & c_3, \\ a_5, & a_7, & a_9, \\ b_5, & b_7, & b_9, \\ c_5, & c_7, & c_9, \\ a_4, & a_6, & a_8, \\ b_4, & b_6, & b_8, \\ c_4, & c_6, & c_8. \end{array}$$

The investigations last given are almost complete in themselves as the geometrical theory of the subject: there is however some difficulty in seeing *à priori* the nature of the correspondence between the planes which determines which are the planes which ought to be distinguished with the same one of the symbolic numbers, 1, 2, 3.

There is great difficulty in conceiving the complete figure formed by the twenty-seven lines, indeed this can hardly I think be accomplished until a more perfect notation is discovered. In the mean time it is easy to find theorems which partially exhibit the properties of the system. For instance, any two lines, a_1, b_2, which do not meet are intersected by five other lines, a_2, b_1, a_5, a_7, a_9, (no two of which meet). Any four of these last-mentioned lines are intersected by the lines a_1, b_2 and no other lines, but any three of them, *e.g.* a_5, a_7, a_9, are intersected by the lines a_1, b_2, and by some third line (in the case in question the line c_3). Or generally any three lines, no two of which meet, are intersected by three other lines, no two of which meet. Again, the lines which do not meet any one of the lines a_5, a_7, a_9, are a_2, a_3, b_3, b_1, c_1, c_2—these lines forming a hexagon, the pairs of the opposite sides of which, a_2, b_1; a_3, c_1; b_3, c_2, are met by the pairs a_1, b_2; c_3, a_1 and a_1, b_2, respectively, viz. by pairs

out of the system of three lines intersecting the system a_5, a_7, a_9. And the lines a_5, a_7, a_9 may be considered as representing any three lines no two of which meet. Again, consider three lines in the same treble tangent plane, *e.g.* a_1, b_1, c_1, and the hexahedron formed by any six treble tangent planes passing two and two through these lines, *e.g.* the planes $x, y, z, \xi, \eta, \zeta$. These planes contain (independently of the lines a_1, b_1, c_1) the twelve lines $a_2, a_3, a_4, a_5, b_2, b_3, b_4, b_5, c_2, c_3, c_4, c_5$. Consider three contiguous faces of the hexahedron, *e.g.* x, y, z, the lines in these planes, viz. $a_4, b_5, c_4, a_5, b_4, c_5$, form a hexagon the opposite sides of which intersect in a point, or in other words these six lines are generating lines of a hyperboloid. The same property holds for the systems x, η, ζ; ξ, y, ζ; ξ, η, z. But for the system ξ, η, ζ, the six lines are a_2, b_2, c_2, and a_3, b_3, c_3, which form two triangles, and similarly for the systems ξ, y, z; x, η, z; and x, y, ζ; so that the twelve lines form four hexagons (the opposite sides of which intersect) circumscribed round four of the angles of the hexahedron, and four pairs of triangles about the opposite four angles of the hexahedron. The number of such theorems might be multiplied indefinitely, and the number of different combinations of lines or planes to which each theorem applies is also very considerable.

Consider the four planes x, ξ, x, $\bar{\mathrm{x}}$, and represent for a moment the equations of these planes by

$$x + Aw = 0, \quad x + Bw = 0, \quad x + Cw = 0, \quad x + Dw = 0,$$

so that

$$A = 0, \ B = \frac{1}{k}\left(m - \frac{1}{m}\right)\left(n - \frac{1}{n}\right), \ C = \frac{l(p-\alpha) + 2mn}{p+\beta},$$

$$D = \frac{\frac{1}{l}(p-\alpha) + \frac{2}{mn}}{p-\beta}.$$

By the assistance of

$$B - C = \frac{-1}{lmn(p^2 - \beta^2)}[ln(p-\alpha) + 2m][lm(p-\alpha) + 2n],$$

it is easy to obtain

$$\frac{(A-C)(D-B)}{(A-D)(B-C)}$$

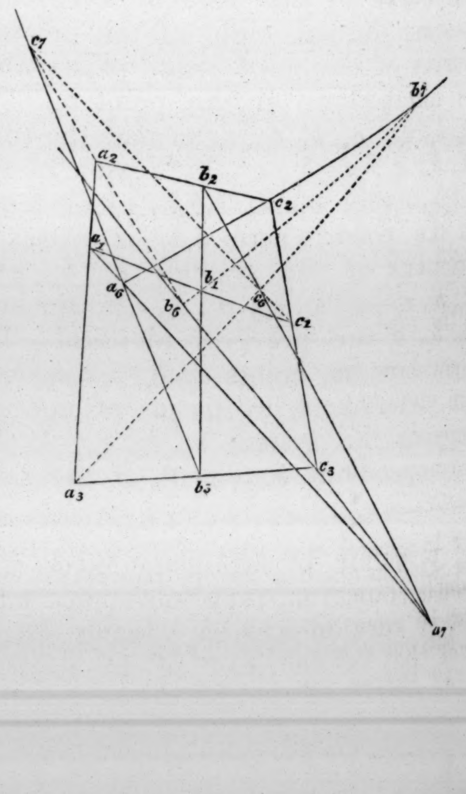
$$= lmn\frac{p-\beta}{p+\beta}\frac{[l(p-\alpha) + 2mn][m(p-\alpha) + 2nl][n(p-\alpha) + 2lm]}{[mn(p-\alpha) + 2l][nl(p-\alpha) + 2m][lm(p-\alpha) + 2n]},$$

which remains unaltered for cyclical permutations of l, m, n, *i.e.* the anharmonic ratio of x, ξ, x, $\bar{\mathrm{x}}$ is the same as that of y, η, y, $\bar{\mathrm{y}}$, or z, ζ, z, $\bar{\mathrm{z}}$; there is of course no correspondence of x to y or ξ to η, &c., the correspondence is by the general properties of anharmonic ratios, a correspondence of the system x, ξ, x, $\bar{\mathrm{x}}$, to any one of the systems $(y, \eta, \mathrm{y}, \bar{\mathrm{y}})$, or $(\eta, y, \bar{\mathrm{y}}, \mathrm{y})$, or $(\mathrm{y}, \bar{\mathrm{y}}, y, \eta)$, or $(\bar{\mathrm{y}}, \mathrm{y}, \eta, y)$, indifferently. The theorems may be stated generally as follows: "Considering two lines in the same treble tangent plane, the remaining treble tangent planes through these two lines respectively are homologous systems."

Suppose the surface of the third order is intersected by an arbitrary plane. The curve of intersection is of course one of the third order, and the positions upon this curve of six of the points in which it is intersected may be arbitrarily assumed. Let these points be the points in which the plane is intersected by the lines a_1, b_1, a_6, b_6, c_6, a_8; or as we may term them, the points a_1, b_1, a_6, b_6, c_6, a_8.* The point c_1 is of course the point in which the line a_1b_1 intersects the curve. The straight lines $a_4b_6c_8$, $b_4c_6a_8$, $c_4a_6b_8$, and $a_4b_8c_6$, $b_4c_8a_6$, $c_4a_8b_6$, show that c_4 and b_4 are the points in which a_8b_6, and a_8c_6 intersect the curve, and then b_8 and c_8 are determined as the intersections with the curve of a_6c_4, a_6b_4. The intersection of the lines b_6c_8 and b_8c_6 (which is known to be a point upon the curve by the theorem, every curve of the third order passing through eight of the points of intersection of two curves of the third order passes through the ninth point of intersection) is the point a_4. The systems a_4, b_4, c_4; a_6, b_6, c_6; a_8, b_8, c_8, determine the conjugate system a_5, b_5, c_5; a_7, b_7, c_7; a_9, b_9, c_9; by reason of the straight lines $a_1a_4a_5$, $b_1b_4b_5$, $c_1c_4c_5$; $a_1a_6a_7$, $b_1b_6b_7$, $c_1c_6c_7$; $a_1a_8a_9$, $b_1b_8b_9$, $c_1c_8c_9$,—*i.e.* a_5 is the point where a_1a_4 intersects the curve, &c. The relations of the systems a_4, b_4, c_4; a_5, b_5, c_5—a_6, b_6, c_6; a_7, b_7, c_7—a_8, b_8, c_8; a_9, b_9, c_9 to the system a_1, b_1, c_1; a_2, b_2, c_2; a_3, b_3, c_3 are precisely identical. It is only necessary to show how the points a_2, b_2, c_2; a_3, b_3, c_3 of the latter system are determined by means of one of the former systems, suppose the system a_6, b_6, c_6; a_7, b_7, c_7; and to discover a compendious statement of the relation between the two

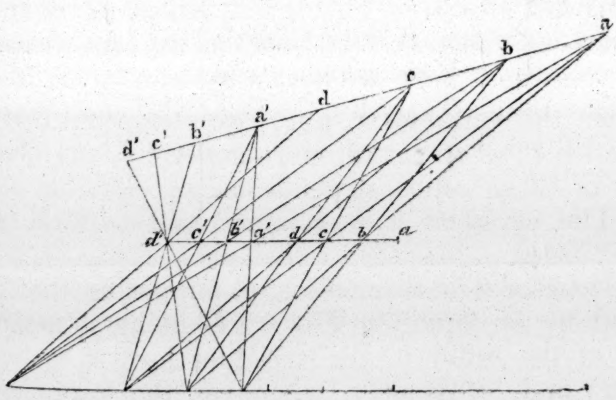
* In general, the point in which any line upon the surface intersects the plane in question may be represented by the symbol of the line, and the line in which any treble tangent plane intersects the plane in question may be represented by the symbol of the treble tangent plane: thus, a_1, b_1, c_1 are points in the line $a_1b_1c_1$ or in the line w, &c.

systems. The points a_1, b_1, c_1; a_2, b_2, c_2; a_3, b_3, c_3; a_6, b_6, c_6; a_7, b_7, c_7, are a system of fifteen points lying on the fifteen straight lines $a_1b_1c_1$, $a_2b_2c_2$, $a_3b_3c_3$, $a_1a_2a_3$, $b_1b_2b_3$, $c_1c_2c_3$, $a_1a_6a_7$, $b_1b_6b_7$, $c_1c_6c_7$, $a_3b_7c_6$, $b_3c_7a_6$, $c_3a_7b_6$, $a_2b_6c_7$, $b_2c_6a_7$, $c_2a_6b_7$, *i.e.* the nine points a_1, b_1, c_1; a_2, b_2, c_2; a_3, b_3, c_3 are the points of intersection of the three lines $a_1b_1c_1$, $a_2b_2c_2$, $a_3b_3c_3$ with the three lines $a_1a_2a_3$, $b_1b_2b_3$, $c_1c_2c_3$, and the remaining six points form a hexagon $a_6b_7c_6a_7b_6c_7$, of which the diagonals a_6a_7, b_6b_7, c_6c_7 pass through the points a_1, b_1, c_1, respectively, the alternate sides a_6b_7, c_6a_7, and b_6c_7 pass through the points c_2, b_2, a_2 respectively, and the remaining alternate sides b_7c_6, a_7b_6, and c_7a_6 pass through the three points a_3, b_3, c_3 respectively. The fifteen points of such a system do not necessarily lie upon a curve of the third order, as will presently be seen: in the actual case however where all the points lie upon a given curve of the third order, and the points a_1, b_1, c_1; a_6, b_6, c_6; a_7, b_7, c_7 are known, a_2, b_2, c_2; a_3, b_3, c_3 are the intersections of the curve with b_6c_7, c_6a_7, a_6b_7, b_7c_6, c_7a_6, a_7b_6 respectively, and the fact of the existence of the lines $a_1b_1c_1$, $a_2b_2c_2$, $a_3b_3c_3$, $a_1a_2a_3$, $b_1b_2b_3$, $c_1c_2c_3$ is an immediate consequence of the theorem quoted above with respect to curves of the third order—a theorem from which the entire system of relations between the twenty-seven points on the curve might have been deduced *à priori*. But returning to the system of fifteen points, suppose the lines $a_1b_1c_1$, $a_2b_2c_2$, $a_3b_3c_3$, and $a_1a_2a_3$, $b_1b_2b_3$, and also the point a_6 to be given arbitrarily. The point a_7 lies on the line a_1a_6, suppose its position upon this line to be arbitrarily assumed (in which case, since the ten points a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, are sufficient to determine a curve of the third order, there is no curve of the third order through these points and the point a_7). If the points b_6, c_6, b_7, c_7 can be so determined that the sides of the quadrilateral $b_6b_7c_6c_7$, viz. b_6b_7,



b_7c_6, c_6c_7, c_7b_6 pass through the points b_1, a_3, c_1, a_2 respectively, while the angles b_6, b_7, c_6, c_7 lie upon the lines a_7c_3, a_6c_2, a_7b_2 and a_6b_3 respectively, the required conditions will be satisfied by the fifteen points in question; and the solution of this problem is known. I have not ascertained whether in the case of an arbitrary position as above of the point a_7, it is possible to determine a complete system of twenty-seven points lying three and three upon forty-five lines in the same manner as the twenty-seven points upon the curve of the third order, but it appears probable that this is the case, and to determine whether it be so or not, presents itself as an interesting problem for investigation.

Suppose that the intersecting plane coincides with one of the treble tangent planes. Here we have a system of twenty-four points, lying eight and eight in three lines; the twenty-four points lie also three and three in thirty-two lines, which last-mentioned lines therefore pass four and four through the twenty-four points. If we represent by $a, b, c, d, a', b', c', d'$—and a, b, c, d, a′, b′, c′, d′, the eight points, and



eight points which lie upon two of the three lines (the order being determinate), the systems of four lines which intersect in the eight points of the third line are

$$(a\mathrm{a},\ b\mathrm{b},\ c\mathrm{c},\ d\mathrm{d}),\quad (a'\mathrm{a}',\ b'\mathrm{b}',\ c'\mathrm{c}',\ d'\mathrm{d}');$$
$$(a\mathrm{b}',\ b\mathrm{a}',\ c'\mathrm{d},\ d'\mathrm{c}),\quad (a'\mathrm{b},\ b'\mathrm{a},\ c\mathrm{d}',\ d\mathrm{c}');$$
$$(a\mathrm{c}',\ c\mathrm{a}',\ d'\mathrm{b},\ b'\mathrm{d}),\quad (a'\mathrm{c},\ c'\mathrm{a},\ b\mathrm{d}',\ d\mathrm{b}');$$
$$(a\mathrm{d}',\ d\mathrm{a}',\ b'\mathrm{c},\ c'\mathrm{b}),\quad (a'\mathrm{d},\ d'\mathrm{a},\ b\mathrm{c}',\ c\mathrm{b}'):$$

the principle of symmetry made use of in this notation (which however represents the actual symmetry of the system very imperfectly) being obviously entirely different from that of the case of an arbitrary intersecting plane,

The transition case where the intersecting plane passes through one of the lines upon the surface (*i.e.* is a double tangent plane) would be worth examining. It should be remarked that the preceding theory is very materially modified when the surface of the third order has one or more conical points; and in the case of a double line (for which the surface becomes a ruled surface) the theory entirely ceases to be applicable. I may mention in conclusion that the whole subject of this memoir was developed in a correspondence with Mr. Salmon, and in particular, that I am indebted to him for the determination of the number of lines upon the surface and for the investigations connected with the representation of the twenty-seven lines by means of the letters a, c, e, b, d, f. as developed above.

ON THE ORDER OF CERTAIN SYSTEMS OF ALGEBRAICAL EQUATIONS.*

By Arthur Cayley.

Suppose the variables $x, y\ldots$ so connected that any one of the ratios $x:y:z,\ldots$ or, more generally, any determinate function of these ratios, depends on an equation of the μ^{th} order. The variables $x, y, z\ldots$ are said to form a system of the μ^{th} order.

In the case of two variables x, y, supposing that these are connected by an equation $U=0$ (U being a homogeneous function of the order μ) the variables form a system of the μ^{th} order; and, conversely, whenever the variables form a system of the μ^{th} order, they are connected by an equation of the above form.

In the case of a greater number of variables, the question is one of much greater difficulty. Thus with three variables x, y, z; if μ be resolvable into the factors μ', μ'', then, supposing the variables to be connected by the equations $U=0$, $V=0$, U and V being homogeneous functions of the orders μ', μ'', respectively, they will it is true form a system of the μ^{th} order, but the converse proposition does not hold: for instance, if μ is a prime number, the only mode of forming

* This memoir was intended to appear at the same time with Mr. Salmon's "Note on a Result of Elimination," (*Journal*, vol. III. p. 109) with which it is very much connected.

a system of the μ^{th} order would on the above principle be to assume $\mu' = \mu$, $\mu'' = 1$, *i.e.* to suppose the variables connected by an equation of the μ^{th} order and a linear equation; but this is far from being the most general method of obtaining such a system. For instance, systems not belonging to the class in question may be obtained by the introduction of subsidiary variables to be eliminated. The simplest example is the following: suppose a, b, a', b', a'', b'' to be linear functions (without constant terms) of x, y, z, and write

$$\begin{cases} a\xi + b\eta = 0, \\ a'\xi + b'\eta = 0, \\ a''\xi + b''\eta = 0; \end{cases}$$

equations from which, by the elimination of ξ, η, two relations may be obtained between the variables x, y, z.

Suppose, however, from these three equations x, y, z are first eliminated: the ratio $\xi : \eta$ will evidently be determined by a cubic equation; and assuming $\xi : \eta$ to be equal to one of the roots of this, any two of the three equations may be considered as implying the third; and will likewise determine linearly the ratios $x : y : z$. Hence any determinate function of these ratios depends on a cubic equation only, or the system is one of the third order. But the order of the system may be obtained by means of the equations resulting from the elimination of ξ, η; and since this will explain the following more general example (in which the corresponding process is the only one which readily offers itself), it will be convenient to deduce the preceding result in this manner. Thus, performing the elimination, we have

$$L = (a'b'' - a''b') = 0, \quad L' = (a''b - ab'') = 0, \quad L'' = (ab' - a'b) = 0.$$

Here the equations $L = 0$, $L' = 0$, $L'' = 0$, are each of them of the second order, and any two of them may be considered as implying the third. For we have identically,

$$aL + a'L' + a''L'' = 0,*$$

so that $L = 0$, $L' = 0$, gives $a''L'' = 0$, or $L'' = 0$. Nevertheless the system is imperfectly represented by means of two equations only. For instance, $L = 0$, $L' = 0$ do, of themselves, represent a system which is really of the fourth

* Also $bL + b'L' + b''L'' = 0$: but since by the elimination of L'', taking into account the actual values of L and L', we obtain an identical equation, these two relations may be considered as equivalent to a single one.

order. In fact, these equations are satisfied by $a''=0, b''=0$, (which is to be considered as forming a system of the first order), but these values do not satisfy the remaining equation $L''=0$. In other words, the equations $L=0, L'=0$ contain an extraneous system of the first order, and which is seen to be extraneous by means of the last equation L'': the system required is the system of the third order which is common to the three equations $L=0, L'=0, L''=0$.

Suppose, more generally, x, y, z are connected by $\overline{p+1}$ equations, involving p variables $\xi, \eta, \zeta\ldots$,

$$\begin{aligned} a\xi + b\eta + c\zeta + \ldots &= 0, \\ a'\xi + b'\eta + c'\zeta + \ldots &= 0. \\ \vdots \end{aligned}$$

Or what comes to the same by the equations (equivalent to two independent relations)

$$\left\| \begin{matrix} a, & a', & a'', & \ldots & a^{(p)} \\ b, & b', & b'', & \ldots & b^{(p)} \\ \vdots \end{matrix} \right\| = 0;$$

(where the number of horizontal rows is p). Consider x, y, z, as connected by the two equations

$$\left| \begin{matrix} a & \ldots & a^{(p-2)}, & a^{(p-1)} \\ b & \ldots & b^{(p-2)}, & b^{(p-1)} \\ \vdots \end{matrix} \right| = 0, \quad \left| \begin{matrix} a & \ldots & a^{(p-2)}, & a^{p} \\ b & \ldots & b^{(p-2)}, & b^{p} \\ \vdots \end{matrix} \right| = 0:$$

these form a system of the order p^2, but they involve the extraneous system

$$\left\| \begin{matrix} a, & b \ldots\ldots \\ \vdots \\ a^{(p-2)} & b^{(p-2)} \ldots \end{matrix} \right\| = 0.$$

Suppose $\phi(p)$ is the order of the system in question, then the order of this last system is $\phi(p-1)$ and hence $\phi p = p^2 - \phi(p-1)$: observing that $\phi(2)=3$, this gives directly $\phi p = \frac{1}{2}p(p+1)$. Hence the order of the system is $\frac{1}{2}p(p+1)$.

Suppose x, y, z, connected by equations of the form $U=0$, $V=0$, $W=0$; U, V, W being linear in x, y, z, and homogeneous functions of the orders m, n, p respectively in ξ, η. By eliminating x, y, z, the ratio $\xi:\eta$ will be determined by an equation of the order $m+n+p$; and since when this is known the ratios $x:y:z$ are linearly

determinable, we have $m+n+p$ for the order μ of the system.

Thus, if $m=n=p=2$, selecting the particular system

$$a\xi^2 + 2b\xi\eta + c\eta^2 = 0,$$
$$b\xi^2 + 2c\xi\eta + d\eta^2 = 0,$$
$$c\xi^2 + 2d\xi\eta + e\eta^2 = 0,$$

it is possible in this case to obtain two resulting equations of the orders two and three respectively, and which consequently constitute the system of the sixth order, without containing any extraneous system. In fact, from the identical equation

$$\begin{aligned} &e\xi.(a\xi^2 + 2b\xi\eta + c\eta^2) \\ &- (4d\xi + 2e\eta)(b\xi^2 + 2c\xi\eta + d\eta^2) \\ &+ (3c\xi + 2d\eta)(c\xi^2 + 2d\xi\eta + e\eta^2) \\ &\qquad = (ae - 4bd + 3c^2)\,\xi^3, \end{aligned}$$

and

$$\begin{aligned} &(ce - d^2)(a\xi^2 + 2b\xi\eta + c\eta^2) \\ &+ (cd - be)(b\xi^2 + 2c\xi\eta + d\eta^2) \\ &+ (bd - c^2)(c\xi^2 + 2d\xi\eta + e\eta^2) \\ &\qquad = (ace - ad^2 - eb^2 - c^3 + 2bcd)\,\xi^2, \end{aligned}$$

we deduce

$$\begin{cases} ae - 4bd + 3c^2 = 0, \\ ace - ad^2 - eb^2 - c^3 + 2bcd = 0\,; \end{cases}$$

which form the system in question, and may for shortness be represented by $I=0$, $J=0$.

The three equations in ξ, η may be considered as expressing that

$$a\xi^3 + 3b\xi^2\eta + 3c\xi\eta^2 + d\eta^3 = 0,$$
$$b\xi^3 + 3c\xi^2\eta + 3d\xi\eta^2 + e\eta^3 = 0,$$

have a pair of equal roots in common; in other words, that it is possible to satisfy identically

$$\begin{aligned} &(A\xi + B\eta)(a\xi^3 + 3b\xi^2\eta + 3c\xi\eta^2 + d\eta^3) \\ &+ (A'\xi + B'\eta)(b\xi^3 + 3c\xi^2\eta + 3d\xi\eta^2 + e\eta^3) = 0. \end{aligned}$$

Equating to zero the separate terms of this equation, and eliminating A, B, A', B', we obtain

$$\left\| \begin{matrix} . & a & 3b & 3c & d \\ . & b & 3c & 3d & e \\ a & 3b & 3c & d & . \\ b & 3c & 3d & e & . \end{matrix} \right\| = 0.$$

It is not at first sight obvious what connection these equations have with the two, $I=0$, $J=0$, but by actual expansion they reduce themselves to the following five,

$$\begin{aligned} 3.[2(ce-d^2)\,I-3eJ]&=0,\\ 3.[(be-cd)\,I-3dJ]&=0,\\ -(ae+2bd-3c^2)\,I+9cJ&=0,\\ 3\,[(ad-bc)\,I-3bJ]&=0,\\ 3\,[2(ac-b^2)\,I-3aJ]&=0; \end{aligned}$$

which are satisfied by $I=0$, $J=0$. By the theorem above given, the equations are to be considered as forming a system of the tenth order; the system must therefore be considered as composed of the system $I=0$, $J=0$, and of a system of the fourth order. The system of the fourth order may be written in the form

$$\begin{array}{lccccccccccc} & 2(ac-b^2) & : & ad-bc & : & ae+2bd-3c^2 & : & be-cd & : & 2ce-d^2 & : & 3J \\ = & a & : & b & : & 3c & : & d & : & e & : & I: \end{array}$$

but to justify this, it must be shewn first that these equations reduce themselves to two independent equations; and next that system is really one of the fourth order. We may remark in the first place, that if

$$u=a\xi^4+4b\xi^3\eta+6c\xi^2\eta^2+4d\xi\eta^3+e\eta^4$$

is a perfect square, the coefficients will be proportional to those of $\dfrac{d^2u}{d\xi^2}\cdot\dfrac{d^2u}{d\eta^2}-\left(\dfrac{d^2u}{d\xi d\eta}\right)^2$.* Thus the conditions requisite in order that u may be a perfect square, are given by the system

$$\begin{array}{lccccccccc} & 2(ac-b^2) & : & (ad-bc) & : & ae+2bd-3c^2 & : & be-cd & : & 2(ce-d^2) \\ = & a & : & b & : & 3c & : & d & : & e. \end{array}$$

Or these equations are equivalent to two independent equations only (this may be easily verified *à posteriori*); and by writing $3J$ in the form

$$e(ac-b^2)-2d(ad-bc)+c(ae+2bd-3c^2)-2b(be-cd)+a(ce-d^2),$$

the remaining equations of the complete system (3) are

* More generally whatever be the order of u, if u contain a square factor, this square factor may easily be shewn to occur in $\dfrac{d^2u}{d\xi^2}\cdot\dfrac{d^2u}{d\eta^2}-\left(\dfrac{d^2u}{d\xi d\eta}\right)^2$,

immediately deduced; *i.e.* this latter system contains only two independent equations. (The preceding reasoning shows that the system (3) expresses the conditions in order that the equations

$$a\xi^3 + 3b\xi^2\eta + 3c\xi\eta^2 + d\eta^3 = 0, \quad b\xi^3 + 3c\xi^2\eta + 3d\xi\eta^2 + e\eta^2 = 0,$$

may have a pair of unequal roots in common: we have already seen that the equations $I = 0$, $J = 0$ represent the conditions in order that these two equations may have a pair of equal roots in common.) Finally, to verify *à posteriori* the fact of the system (3) being one only of the fourth order, we may, as Mr. Salmon has done in the memoir above referred to, represent the system by the two equations

$$a(ce - d^2) - e(ac - b^2) = 0, \quad e(ad - bc) - 2b(ce - d^2) = 0,$$

i.e. by $\quad ad^2 - eb^2 = 0, \quad 2bd^2 - 3bce + ade = 0.$

These equations contain the extraneous system $(a = 0,\ b = 0)$ and the extraneous systems $(b = 0,\ d = 0)$ and $(d = 0,\ e = 0)$, each of which last, as Mr. Salmon has remarked from geometrical considerations, counts double, or the system is one of the 4th order only.

ON A POINT IN THE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS.

By Augustus De Morgan, &c.

I can hardly tell whether the following simple theorem is or is not to be collected from existing writings: but in any case it has not the prominence it ought to have.

If U and V be two particular solutions of a linear differential equation, and if, a and b being two *uneliminated** constants, $a = \alpha$ and $b = \beta$ give $U = V$, then $AU + BV$, the part of the solution which usually restores two eliminated constants, restores only one. Let $a = \alpha + \theta$, $b = \beta + \theta$, and

* Meaning, which appear in the differential equation, and which are eliminated in *other* differential equations formed from the same primitive. The usual mode of distinction is that of *given* and *arbitrary*, which is founded on the differential equation, apart from the consideration of the problem which yields it. Beginners are much confused when, as is frequently the case, one of the uneliminated constants is arbitrary, and one of the eliminated constants given or determinable.

let the first differential coefficients of U and V with respect to θ, which do not become identical when $\theta = 0$, be of the k^{th} order. Throw $AU + BV$ into the form

$$(A + B)\ U + B\theta^k \{(V - U) \div \theta^k\},$$

and, remembering that B and A may be functions of θ, let $A + B = A'$, $B\theta^k = 2.3\ldots k.B'$, A' and B' being new constants. Then, making $\theta = 0$, we have, for a solution with two restored constants,

$$A'U + B'\left(\frac{d^kV}{db^k} - \frac{d^kU}{da^k}\right) \quad (a = \alpha,\ b = \beta).$$

If three particular solutions, U, V, W, merge in one when $a = \alpha$, $b = \beta$, $c = \gamma$, we have, by a similar process,

$$A'U + B'\left(\frac{d^kV}{db^k} - \frac{d^kU}{da^k}\right) + C'\left(\frac{d^lW}{dc^l} - \frac{d^lU}{da^l}\right) (a = \alpha,\ b = \beta,\ c = \gamma).$$

If two reduced solutions agree with one another, or one reduced solution with any particular solution of another kind, the same process is to be employed on the forms which precede the assumptions $a = \alpha$, &c. Thus

$$\{x^2 + (1 + a)\ cx\}\ y'' - (1 + a)(x + ac)\ y' + (1 + a)\ y = 0$$

is solved by $y = Ax^{1+a} + B(x + ac)$, which, when $a = 0$, becomes $Ax + Bx$. The defective case is supplied by making $a = 0$ in $x^{1+a} \log x - c$, and the complete solution of

$$(x^2 + cx)\ y'' - xy' + y = 0 \text{ is } y = Ax + B(x \log x - c).$$

There is one apparent case of exception, namely when U, V, &c. are of the form $\phi(a, x)$, $\phi(b, x)$, &c., so that $a = \alpha$, $b = \alpha$, makes U and V, and all differential coefficients identical. But if, in such a case, we begin with $\phi(a, x)$, and $m\,\phi(b, x)$ instead of $\phi(b, x)$, we have the loss of a solution when $a = b$, and the differential coefficients not identical. The result will immediately follow that $(1 - m)\dfrac{dU}{da}$, and therefore $\dfrac{dU}{da}$, is a solution.

In this manner all that is commonly given on the loss of solutions by particular coincidence of generally different forms can be reduced to one method. Writers seem to have been dissatisfied with all methods of treating these cases except those which preserve the differences of the constants perfectly general until they vanish. But *any* process which gives as many distinct solutions as there can be constants

yields the most general solution. The common case, for example, in which coincidence of constants produces identity of value where before there was only sameness of form, is most easily treated thus. Let T, U, V, W, be four solutions which are of the form $\phi(a, x)$, $\phi(b, x)$, &c. Let b, c, e be $a+\theta$, $a+2\theta$, $a+3\theta$, and throw $AT+BU+CV+EW$ into the form

$$A'T+B'\Delta T+C'\Delta^2 T+E'\Delta^3 T,$$

ΔT being $U-T$, &c.; and this again into the form

$$A'+B'\theta\,\frac{\Delta T}{\theta}+C'\theta^2\,\frac{\Delta^2 T}{\theta^2}+E'\theta^3\,\frac{\Delta^3 T}{\theta^3},$$

whence the usual result is easily obtained.

ON THE RELATION BETWEEN DIFFERENT CURVES AND CONES CONNECTED WITH A SERIES OF CONFOCAL ELLIPSOIDS.

By John Y. Rutledge, B.A., Trinity College, Dublin.

The object of the present paper is to shew the natural connexion that exists between several systems of curves traced upon the surface of an ellipsoid, which hitherto have been viewed as isolated objects of curiosity, but which, from the fact of their resulting from one and the same source, may be regarded in a still more interesting and instructive point of view.

It is well known that all questions relating to conjugate diameters in central surfaces of the second order may be discussed as questions relating to rectangular axes, by the transformation

$$\frac{x}{a}=\cos\alpha,\qquad \frac{y}{b}=\cos\beta,\qquad \frac{z}{c}=\cos\gamma,$$

$$\frac{x'}{a}=\cos\alpha',\qquad \frac{y'}{b}=\cos\beta',\qquad \frac{z'}{c}=\cos\gamma',$$

$$\frac{x''}{a}=\cos\alpha'',\qquad \frac{y''}{b}=\cos\beta'',\qquad \frac{z''}{c}=\cos\gamma'',$$

where $x, y, z, x', y', z', x'', y'', z''$, are the coordinates of the extremities of the conjugate semidiameters, and $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$, &c. are the angles made by the new rectangular

axes of reference with the original axes of x, y, z. Suppose that we have a number of confocal ellipsoids:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\quad \frac{x^2}{a'^2}+\frac{y^2}{b'^2}+\frac{z^2}{c'^2}=1, \text{ \&c.}\quad \frac{x^2}{a''^2}+\frac{y^2}{b''^2}+\frac{z^2}{c''^2}=1, \text{ \&c.}$$

In the first let us take three conjugate semidiameters, the respective coordinates of the extremities of which are x, y, z, x', y', z', and x'', y'', z''. Let us take the corresponding points on the remaining ellipsoids, *i.e.* points such that

$$\frac{x}{a}=\frac{x_{,}}{a'}=\frac{x_{,,}}{a''}, \text{ \&c.}$$

Then, if we connect the origin of coordinates with these points on the respective ellipsoids, it is easy to see that we shall have, in each, corresponding sets of conjugate diameters; and that if $r^2 = x^2 + y^2 + z^2$ and $r_{,}^2 = x_{,}^2 + y_{,}^2 + z_{,}^2$, then $r_{,}^2 - r^2 = a'^2 - a^2$, $r_{,,}^2 - r^2 = a''^2 - a^2$, &c. Now if we take the axes of reference, *i.e.* the three rectangular axes to which the three conjugate semidiameters may be referred by the substitution given above; these axes of reference will be the same for the corresponding sets of conjugate diameters in the whole series of confocal ellipsoids; and if we take the tangent planes to the respective ellipsoids, normal to the axes of reference, the lengths of the perpendiculars on the tangent planes will be equal to the respective conjugate semidiameters, *i.e.* $p = r$, $p_{,} = r_{,}$, $p_{,,} = r_{,,}$, &c.

Let us conceive the respective conjugate semidiameters to describe equiradial cones in their respective ellipsoids; we can shew that the corresponding points will group themselves on the sphero-conics thus described upon the surfaces of the ellipsoids, so that we shall obtain sphero-conics as loci of systems of corresponding points.

Now let us turn our attention to the axes of reference. While the equiradial cones are being described, the axes of reference will describe (what we may call) cones of reference, which will be the *same* for the entire series of ellipsoids.

The sides of the (three) equiradial cones in *each* ellipsoid will furnish different sets of conjugate diameters, and for these sides the rectangular axes of reference will form sides of the three cones of reference, which we may therefore call, under this point of view, conjugate cones of reference. In other respects each cone of reference may be viewed separately, and in relation solely to its own set of equiradial cones. Now the perpendiculars upon the tangent planes to each ellipsoid, normal to the sides of each cone of reference,

will be constant; consequently the points of contact will lie upon the curves of constant curvature, the equations of which we will presently obtain. (I mean by curves of constant curvature, curves which pass through the points upon the ellipsoid, at which the *measure* of curvature is constant, *i.e.* where the reciprocal of the product of the two principal radii of curvature is constant.) We thus obtain upon each ellipsoid three curves naturally connected; viz. the sphero-conic, the curve of constant curvature, and the curve in which the cone of reference intersects the ellipsoid, or rather three cones having their vertex at the centre of the ellipsoid, viz. the equiradial cone, the cone through the line of constant curvature, and the cone of reference.

To connect each side of the cone of reference with its correlative sides, on the equiradial cone and the cone through the line of constant curvature, we obtain a simple construction. If the equation of the ellipsoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

let us construct a new ellipsoid, which we may call the ellipsoid of reference, viz.

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1.$$

To this ellipsoid let us draw a tangent plane parallel to the tangent plane to the original ellipsoid and normal to the given axis of reference; through the point of contact of this plane with the ellipsoid of reference draw a line from the centre: it will be the required side of the equiradial cone. Again, normal to this line let us draw a plane tangent to the ellipsoid of reference; join the point of contact with the centre: it will be the side of the cone through the line of constant curvature.

Of course the ellipsoid of reference will be different for each ellipsoid of the series of confocal ellipsoids. If p be the perpendicular upon the tangent plane to the ellipsoid of reference, and ξ, η, ζ the coordinates of the point of contact, $p\xi = x$, $p\eta = y$, $p\zeta = z$, where x, y, z are the coordinates of a point upon the original ellipsoid from which the ellipsoid of reference is derived. The original ellipsoid is the reciprocal polar of the sphere $x^2 + y^2 + z^2 = 1$ with respect to the ellipsoid of reference

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1.$$

It is well known that the equation of the equiradial cone is

$$\frac{a^2 - r^2}{a^2} x^2 + \frac{b^2 - r^2}{b^2} y^2 + \frac{c^2 - r^2}{c^2} z^2 = 0,$$

concyclic with the given ellipsoid: we may find as the equation of the cone of reference

$$(a^2 - r^2) x^2 + (b^2 - r^2) y^2 + (c^2 - r^2) z^2 = 0,$$

concyclic with the reciprocal of the given ellipsoid, and for the cone of the line of constant curvature

$$\frac{a^2 - r^2}{a^4} x^2 + \frac{b^2 - r^2}{b^4} y^2 + \frac{c^2 - r^2}{c^4} z^2 = 0.$$

In which r indicates either the constant side of the equiradial cone, or the constant perpendicular upon the tangent plane: by varying r we get the different curves of constant curvature on the surface of the ellipsoid. If we project this curve upon the planes of circular section by lines parallel to the greatest or least axis of the ellipsoid, the projection will be a conic whose eccentricity is that of the principal section of the ellipsoid perpendicular to that axis, as Mr. Allman has noticed in a recent number of this *Journal*.

The tangent planes to this curve will describe by their ultimate intersections a developable surface. If the ellipsoid be one of revolution (round its mean axis suppose), we find for its equation,

$$\frac{x^2}{a^2 - r^2} + \frac{y^2}{b^2 - r^2} + \frac{z^2}{a^2 - r^2} - 2y \frac{\sqrt{(a^2 - b^2)}}{(b^2 - r^2)\sqrt{(a^2 - r^2)}} = \frac{(a^2 - b^2) r^2}{(a^2 - r^2)(r^2 - b^2)}.$$

The parallel tangent planes, through the centre, will describe a cone whose equation is

$$\frac{x^2}{a^2 - r^2} + \frac{y^2}{b^2 - r^2} + \frac{z^2}{a^2 - r^2} = 0,$$

and in general it will be

$$\frac{x^2}{a^2 - r^2} + \frac{y^2}{b^2 - r^2} + \frac{z^2}{c^2 - r^2} = 0,$$

as it evidently ought to be, for it is the reciprocal of the cone of reference. The focal lines of this cone are the asymptotes of the focal hyperbola, and consequently, as each side of a cone makes angles with the focal lines whose sum or difference is constant, each 'arete' of the developable surface will make angles with the asymptotes of the focal hyperbola,

whose sum or difference is constant, which Mr. Allman has also anticipated me in noticing in the number referred to.

The ultimate intersections of the tangent planes to the ellipsoid along the sphero-conic will describe a developable surface, and the parallel central sections will envelope a cone whose sides are parallel to the 'aretes' of the developable. The equation of this cone is

$$\frac{x^2}{a^2(a^2-r^2)} + \frac{y^2}{b^2(b^2-r^2)} + \frac{z^2}{c^2(c^2-r^2)} = 0,$$

which we recognize as being the curve of intersection of the two confocal surfaces

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad \frac{x^2}{a^2-r^2} + \frac{y^2}{b^2-r^2} + \frac{z^2}{c^2-r^2} = 1,$$

and therefore a line of curvature. Hence we have the theorem—That the cone whose sides are parallel to the 'aretes' of a developable circumscribed to an ellipsoid along a sphero-conic, passes through the line of curvature on the ellipsoid made by the intersection of the ellipsoid with the hyperboloid, whose equation is

$$\frac{x^2}{a^2-r^2} + \frac{y^2}{b^2-r^2} + \frac{z^2}{c^2-r^2} = 1,$$

where r is the sphere radius of the sphero-conic. Again, the cone whose sides are parallel to the 'aretes' of the developable circumscribed to the ellipsoid along this line of curvature, will be the equiradial cone whose radius is r; and hence we have a singularly simple proof of Sir William Hamilton's elegant theorem, "That the tangent plane along a line of curvature on an ellipsoid makes angles with the planes of circular section, whose sum or difference is constant." For, each tangent plane is parallel to a central section tangent to the equiradial cone; but each tangent plane to a cone makes angles with the planes of circular section whose sum or difference is constant. Q. E. D.

We can with equal facility prove that the tangent plane to the ellipsoid along the curve, 'correlative' to the line of curvature, whose equation is

$$\frac{x^2}{a^4.(a^2-r^2)} + \frac{y^2}{b^4.(b^2-r^2)} + \frac{z^2}{c^4.(c^2-r^2)} = 0,$$

makes angles whose sum or difference is constant with the planes normal to the asymptotes of the focal hyperbola.

$a^2 - r^2$, $b^2 - r^2$, $c^2 - r^2$, are not the semimajor axes of the confocal hyperboloid which passes through the point x, y, z, upon the ellipsoid: were that the case, then

$$\frac{x^2}{a^4.(a^2 - r^2)} + \frac{y^2}{b^4.(b^2 - r^2)} + \frac{z^2}{c^4.(c^2 - r^2)} + \frac{1}{P^2 r^2} = 0,$$

where P is the perpendicular from the centre upon the plane tangent to the ellipsoid at any point upon the line of curvature. This last remark is due to Mr. Willock.

When we consider the two remaining conjugate semi-diameters, we obtain two analogous cones and lines of curvature. The equations of the hyperboloids will be

$$\frac{x^2}{a^2 - r'^2} + \frac{y^2}{b^2 - r'^2} + \frac{z^2}{c^2 - r'^2} = 1, \quad \frac{x^2}{a^2 - r''^2} + \frac{y^2}{b^2 - r''^2} + \frac{z^2}{c^2 - r''^2} = 1.$$

Now since $a^2 - r^2 = a'^2 - r_{,}^2 = a''^2 - r_{,,}^2$, &c., these three surfaces remain the same for the entire series of confocal ellipsoids, and may therefore be regarded as three fixed surfaces. The lines of curvature made by their intersection with each ellipsoid of the series can be easily proved to be the loci of systems of corresponding points: we have already seen that the related sphero-conics are similar loci. If we select any two of these three surfaces, and seek what relation they bear to the third semidiameter, we obtain the following. "The locus of the points of rectangular intersection of the geodesic lines, tangent to the lines of curvature made by the intersection of the two hyperboloids with the given ellipsoid, will be the sphero-conic traced by the third (conjugate) semidiameter upon the surface of the ellipsoid." Also, if we select the asymptotic cones of any two of these three surfaces, and look for the locus of the rectangular intersection of their tangent planes, we shall find for its equation

$$(a^2 - r^2)\, x^2 + (b^2 - r^2)\, y^2 + (c^2 - r^2)\, z^2 = 0,$$

which is the cone of reference, corresponding to the third conjugate semidiameter. Its reciprocal, of course, will be the asymptotic cone of the remaining fixed surface.

At the point x, y, z upon the original ellipsoid, let us take two confocal surfaces, whose equations are

$$\frac{x^2}{\mu^2} + \frac{y^2}{\mu'^2} + \frac{z^2}{\mu''^2} = 1 \ (\phi), \quad \frac{x^2}{\rho^2} + \frac{y^2}{\rho'^2} + \frac{z^2}{\rho''^2} = 1 \ (\omega).$$

Let the same conjugate semidiameter r, terminating in the point x, y, z describe in them sphero-conics; by constructing their surfaces of reference we shall get, as before, their cones

of reference and cones of lines of constant curvature. Their cones, whose sides are parallel to the aretes of their developable surfaces, made by the ultimate intersections of the tangent planes along the lines of constant curvature, will be confocal with the cone found for the ellipsoid, and all conclusions will hold as before.

Now if we take the three axes of reference corresponding to the semidiameter r, and draw their normal tangent planes, the points of contact with each surface will be corresponding points, *i.e.*

$$\frac{x_{,}}{a} = \frac{x_{,}'}{\mu} = \frac{x_{,,}}{\rho}, \text{ \&c.}$$

If we consider the corresponding points to x, y, z on the series of confocal ellipsoids; it is well known that the curve of intersection of ϕ and ω will be the locus of these corresponding points; consequently, if we repeat the preceding construction for each corresponding point on the several ellipsoids of the series, the points of contact of the several tangent planes to the surfaces ϕ and ω will give us the points on the surfaces ϕ and ω which correspond to each other.

The cone of the line of constant curvature may be regarded as the asymptotic cone of the surface, whose equation is

$$\frac{a^2 - r^2}{a^4} x^2 + \frac{b^2 - r^2}{b^4} y^2 + \frac{c^2 - r^2}{c^4} z^2 = 1.$$

This surface is the reciprocal polar, with respect to the original ellipsoid of the surface

$$\frac{x^2}{a^2 - r^2} + \frac{y^2}{b^2 - r^2} + \frac{z^2}{c^2 - r^2} = 1,$$

which is one of the fixed surfaces already noticed. Hence we have a very simple construction for the lines of constant curvature on the whole series of ellipsoids. Let the constant perpendicular, upon the tangent plane to the first, be r; upon that to the second r', &c., while $r'^2 - r^2 = a'^2 - a^2$, &c. Then the cones of the lines of constant curvature upon each ellipsoid of the series are given as the asymptotic cones of one and the same surface,

$$\frac{x^2}{a^2 - r^2} + \frac{y^2}{b^2 - r^2} + \frac{z^2}{c^2 - r^2} = 1,$$

taken with respect to each ellipsoid of the series. Let $r^2 = b^2$. Then the preceding cone will become for the first ellipsoid,

$$\frac{a^2 - b^2}{a^4} x^2 + \frac{c^2 - b^2}{c^4} z^2 = 0,$$

the equation to two right lines; at the points in which they pierce the surface, the measure of curvature will be equal to the square of the reciprocal of the perpendicular upon the tangent plane at the umbilic. We can now perceive how, from one and the same surface taken in connexion with the ellipsoid, spring the different curves which have been already enumerated.

Suppose we have any hyperboloid,

$$\frac{x^2}{a^2 - r^2} + \frac{y^2}{b^2 - r^2} + \frac{z^2}{c^2 - r^2} = 1,$$

where r is any semidiameter of the ellipsoid. The intersection of this hyperboloid with the ellipsoid gives us the line of curvature. The cone whose sides are parallel to the 'aretes' of the developable circumscribed to the ellipsoid along this line of curvature, will be the equiradial cone whose sphere-radius $= r$, and will consequently give us the spheroconic. The reciprocal cone will, of course, be the cone of the wave surface. The asymptotic cone of the reciprocal polar of this hyperboloid, taken with respect to the ellipsoid, will be the cone of the line of constant curvature; and the central cone, whose sides are parallel to the 'aretes' of the developable circumscribed to the ellipsoid along this line of constant curvature, is the asymptotic cone of the given hyperboloid; the reciprocal cone of which is the cone of reference corresponding to the equiradial cone. When viewed through the medium of corresponding points, as we have indicated, the same fixed surface will give the analogous related curves upon the infinite series of confocal ellipsoids.

The surface, whose asymptotic cone is the central cone which passes through the line of constant curvature, is the locus of the different 'aretes de rebrousement' of the developable surfaces circumscribed to the given ellipsoid, along the geodesic lines which envelope the line of curvature made by the intersection of the fixed hyperboloid with the given ellipsoid, while the planes, tangent to the fixed hyperboloid and normal to the given ellipsoid, will trace upon the surface of the ellipsoid by the normal points the geodesic lines just mentioned. Could we so ascertain the equation of the geodesic line, as to shew the central cone, whose sides pass through the geodesic line, to be connected with the equiradial cone or with any of the cones which we have indicated, our theory would be complete. It is also important to remember that the consideration of the two semidiameters

of the ellipsoid, conjugate with the semidiameter r, gives us the preceding curves and cones in sets, each set consisting of three; so that any one of the above-mentioned curves or cones being given for the ellipsoid, we have also given with it two analogous conjugate curves or cones.

The following remark may perhaps interest.

An ellipsoid being given, we see that we are giving with it its ellipsoid of reference and the sphere whose radius is unity. The reciprocal polar of any fixed hyperboloid, with respect to the sphere, will be the surface whose asymptotic cone is the cone of reference. Its reciprocal polar, with respect to the ellipsoid of reference, will be the surface whose asymptotic cone is the equiradial cone, and its reciprocal polar with respect to the given ellipsoid will be the surface, as we have frequently remarked, whose asymptotic cone is the cone through the line of constant curvature; while the reciprocal polar of the ellipsoid itself, with respect to the surface of reference of the fixed hyperboloid, will be the surface whose asymptotic cone is the central cone of the wave surface. So that, in fact, the reciprocal polars of the fixed hyperboloid with respect to the ellipsoid of reference, and of the ellipsoid with respect to the hyperboloid of reference, are reciprocals polars, one of the other, with respect to the sphere whose radius is unity.

Let a line pass through two focal curves of an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

and be terminated on either side by the surface of the ellipsoid. Let x, y, z be the coordinates of the extremity of this line, we shall find as values for x, y, z,

$$x = \lambda \pm \mu, \quad y = \lambda' \pm \mu', \quad z = \lambda'' \pm \mu'';$$

in which μ, μ', μ'' denote the parts of the values under the radical sign: we shall find

$$\mu = \frac{R^2}{a} \cos \alpha, \quad \mu' = \frac{R^2}{a} \cos \beta, \quad \mu'' = \frac{R^2}{a} \cos \gamma,$$

where R is the semidiameter parallel to the bifocal line, a the semimajor axis of the ellipsoid, and α, β, γ the angles which R makes with the axes of x, y, z.

In the parabola, of which the equation is

$$x = \frac{y^2}{p} + \frac{z^2}{q}, \quad \mu = \frac{R^2}{2} \cos \alpha, \quad \mu' = \frac{R^2}{2} \cos \beta, \quad \mu'' = \frac{R^2}{2} \cos \gamma,$$

where R is the semidiameter of the ellipse

$$1 = \frac{y^2}{p} + \frac{z^2}{q},$$

and parallel to the bifocal line and α, β, γ, as before; let the line terminated on either side by the surface of the ellipsoid touch two confocal surfaces,

$$\frac{x^2}{a_0^{\ 2}} + \frac{y^2}{b_0^{\ 2}} + \frac{z^2}{c_0^{\ 2}} = 1, \quad \frac{x^2}{a_1^{\ 2}} + \frac{y^2}{b_1^{\ 2}} + \frac{z^2}{c_1^{\ 2}} = 1,$$

then

$$\mu = \frac{R^2 \cos \alpha}{abc} \sqrt{(a^2 - a_0^{\ 2})} \sqrt{(a^2 - a_1^{\ 2})}, \quad \mu' = \frac{R^2 \cos \beta}{abc} \sqrt{(a^2 - a_0^{\ 2})} \sqrt{(a^2 - a_1^{\ 2})},$$

$$\mu'' = \frac{R^2 \cos \gamma}{abc} \sqrt{(a^2 - a_0^{\ 2})} \sqrt{(a^2 - a_1^{\ 2})}.$$

Now $4\mu^2 = (x - x')^2$ is the square of the projection of the bifocal line upon the axis of x: similarly $4\mu'^2 = (y - y')^2$ upon the axis of y, &c. From these expressions we at once deduce the elegant values for bifocal lines given by Professor M'Cullagh, viz. in the ellipsoid, calling l the length of the bifocal line,

$$l^2 = 4(\mu^2 + \mu'^2 + \mu''^2) = 4\frac{R^4}{a^2},$$

therefore $(2al) = (2R)^2$. In the parabola

$$l^2 = 4(\mu^2 + \mu'^2 + \mu''^2) = R^4, \quad \therefore l = R^2, \quad \therefore \frac{1}{l} = \frac{\cos^2\beta}{p} + \frac{\cos^2\gamma}{q}.$$

When the line touches two confocal surfaces, then

$$(l \,.\, 2L) = (2R)^2, \text{ where } L = \frac{abc}{\sqrt{(a^2 - a_0^{\ 2})} \sqrt{(a^2 - a_1^{\ 2})}}.$$

The expressions given for μ, μ', μ'', are correlative in a certain sense to an expression used by Professor M'Cullagh. From any point on the surface of the ellipsoid as vertex describe two cones circumscribing two given confocal surfaces, of which the semimajor axes are a_0 and a_1. The two cones will, in general, intersect in four lines which will make equal angles with the axis of the cones normal to the ellipsoid at the assumed point. For the cosine of this angle Prof. M'Cullagh gave the value $\cos \alpha = \dfrac{\sqrt{(a^2 - a_0^{\ 2})} \sqrt{(a^2 - a_1^{\ 2})}}{r r'}$, where r and $r_{\prime}$ are the principal conjugate semidiameters of the central section of the given ellipsoid, made by a plane

parallel to the tangent plane at the assumed point vertex of the cones.

In the ellipsoid for λ, λ', λ'', we get as values,

$$\lambda = R^2\left(\frac{\cos^2\beta x'}{b^2} + \frac{\cos^2\gamma x_{,}}{c^2}\right),$$

$$\lambda' = -R^2\cos\beta\left(\frac{\cos\alpha x'}{a^2} + \frac{\cos\gamma z'}{c^2}\right),$$

$$\lambda'' = -R^2\cos\gamma\left(\frac{\cos\alpha x_{,}}{a^2} + \frac{\cos\beta y_{,}}{b^2}\right);$$

x' and z' are the coordinates of the point in which the bifocal line pierces the focal hyperbola; $x_{,}$, $y_{,}$, where it pierces the focal ellipse.

They are connected by the equation

$$\frac{x'^2 y_{,}^2}{a^2 - b^2} + \frac{x_{,}^2 z'^2}{a^2 - c^2} = y_{,}^2 + z'^2.$$

These expressions may, perhaps, admit of some farther application than that which we have given.*

THE FOCAL GENERATION OF SURFACES OF THE SECOND ORDER.

By the Rev. Wm. A. Willock.

The generation of surfaces of the second order, which it is the aim of this paper to establish, may deserve some consideration, being a generalization of the well-known modular and umbilicar generations from the focal curves. Professor MacCullagh, in his paper "On Surfaces of the Second Order," published in the *Proceedings of the Royal Irish Academy* (vol. II. p. 471), has indicated a method of this kind, and it is by having followed out the suggestions which he has there given that the following results have been obtained.

As in the generation of surfaces from the focal curves the data assumed are the focus directrix, modulus, and directive plane; so likewise in the method here given there are certain analogous data, but different from the above. They are:—

* We have since found that the preceding expressions are the natural foundation of a very beautiful class of theorems originating with Professors Chasles and MacCullagh. The complete development we hope to give upon some future occasion.

A central surface of the second order anywhere placed, and two points.

The points may be called *the focus* and *dirigent,* respectively, on account of the analogies they bear to the focus and directrix in focal curves. The analogy would lead us to extend the name directrix to the second point, but as this word could not well be used to denote *a point,* the name *dirigent* seems preferable.

The surface may be called the *modular surface,* and a central radius of it a *modular radius.* The three *numbers,* real or imaginary, which denote the ratios which some arbitrary line K bears to the three semiaxes may be called the *moduli.* If the line K be the linear unit, the moduli will be the reciprocals of the semiaxes. These moduli may be denoted by the symbols λ, μ, ν, so that the following relations always hold, viz.

$$\lambda = \frac{K}{A}, \quad \mu = \frac{K}{B}, \quad \nu = \frac{K}{C},$$

A, B, and C being the three semiaxes. The reason for calling this surface the modular surface is, that in the use now made of it, it is analogous to the modulus of a focal curve.

These preliminaries being settled, the following principle is true for all surfaces of the second order, *central* and *non-central.*

The locus of a point whose distance from the focus is to its distance from the dirigent as the arbitrary line K to the modular radius which is parallel to the dirigent distance, is a surface of the second order, condirective with the modular surface.

The focus and dirigent are not absolutely fixed, but (the modular and generated surfaces remaining unchanged) capable of varying according to certain laws, viz.—

The locus of the focus is a surface of the second order confocal with the generated surface; and—

The locus of the dirigent is also a surface of the second order, the polar reciprocal of the former with respect to the latter surface.

Each individual focus and its corresponding dirigent are reciprocal points with respect to the generated surface; and at the same time also corresponding points (in the common sense) with respect to their own surfaces.

These surfaces may be called the *focal* and *dirigent surfaces* respectively, being analogous to the focal curve and dirigent cylinder of the modular and umbilicar methods of generation.

The principles contained in the above propositions can be thus established. Let r and ρ denote the focal and dirigent distances, R the modular radius, and α, β, γ the angles made by this radius with the axes; we have then this equation,

$$\frac{r^2}{\rho^2} = \frac{K^2}{R^2} = \frac{K^2}{A^2}\cos^2\alpha + \frac{K^2}{B^2}\cos^2\beta + \frac{K^2}{C^2}\cos^2\gamma,$$

which becomes, by introducing the moduli and denoting the coordinates of the focus and dirigent respectively by x_1, y_1, z_1, x_2, y_2, z_2,

$$(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2 = \lambda^2(x-x_2)^2 + \mu^2(y-y_2)^2 + \nu^2(z-z_2)^2,$$

and gives, for the equation of the required locus,

$$(1-\lambda^2)\,x^2 + (1-\mu^2)\,y^2 + (1-\nu^2)\,z^2$$
$$- 2(x_1 - \lambda^2 x_2)\,x - 2(y_1 - \mu^2 y_2)\,y - 2(z_1 - \nu^2 z_2)\,z = H,$$

where H denotes the quantity

$$(\lambda^2 x_2^{\,2} + \mu^2 y_2^{\,2} + \nu^2 z_2^{\,2}) - (x_1^{\,2} + y_1^{\,2} + z_1^{\,2}).$$

Now if in this equation we denote the quantities

$$\frac{x_1 - \lambda^2 x_2}{1-\lambda^2}, \quad \frac{y_1 - \mu^2 y_2}{1-\mu^2}, \quad \frac{z_1 - \nu^2 z_2}{1-\nu^2},$$

by l, m, n it may be thrown into the following convenient form:

$$(1)\ (1-\lambda^2)(x^2 - 2lx) + (1-\mu^2)(y^2 - 2my) + (1-\nu^2)(z^2 - 2nz) = H,$$

from which its properties may more easily be derived.

The equations of the focal and dirigent surfaces can by the elimination of either set of coordinates, by means of the values of l, m, and n, from the expression for H, be shewn to be of the form

$$(2)\ \frac{(1-\lambda^2)}{\lambda^2}(x_1^{\,2} - 2lx_1) + \frac{(1-\mu^2)}{\mu^2}(y_1^{\,2} - 2my_1) + \frac{(1-\nu^2)}{\nu^2}(z_1^{\,2} - 2nz_1) = H_1,$$

$$(3)\ (1-\lambda^2)\,\lambda^2(x_2^{\,2} - 2lx_2) + (1-\mu^2)\,\mu^2(y_2^{\,2} - 2my_2)$$
$$+ (1-\nu^2)\,\nu^2(z_2^{\,2} - 2nz_2) = H_2.$$

From a comparison of which with equation (1) we can infer—

1. They have the same centre whether at a finite or infinite distance. This appears from the quantities l, m, n, (which are the coordinates of the centre of the first) entering after the same manner in the three equations.

2. The generated surface is condirective with the modulus, for the differences of the coefficients of the squares of the variables in its equation are equal to the corresponding differences in the equation of the modular surface.

3. The focal and generated surfaces are confocal; for it can be shewn, by transferring the origin of coordinates to the common centre, that the absolute terms in the equations of the three surfaces will be equal.* This combined with the consideration that the asymptotic cones of (1) and (2) are confocal, shews that the surfaces are also confocal.

4. Surface (3) is the polar reciprocal of (2) with respect to (1); for, there being a common centre, the coefficients of the squares of the variables being in geometric progression, and the absolute terms now mentioned being equal, these are sufficient to indicate the property in question.

These principles are general, and irrespective of the particular nature of the surfaces. It will further be necessary to consider them according as they are central or non-central. This division is here derived from a consideration of the moduli. When the moduli are all greater or less than *unity*, the surfaces have finite axes; but when they are taken, one or more of them, equal to this number, the corresponding axes become infinite. Hence the following classification, in which it is to be carefully observed that the moduli, from the nature of the case, must be considered as all essentially *positive,* though they may be either *real or imaginary.*

I. Moduli all different from unity; central surfaces containing the varieties—

(A) Three moduli unequal; ellipsoids and hyperboloids of both species, including cones.

(B) Two moduli equal; surfaces of revolution, including right cones.

(C) Three moduli equal; spheres only real and imaginary.

II. Moduli equal unity; non-central surfaces containing the varieties—

(A) One modulus unity and other two unequal; paraboloids, including cylinders and the particular case of pairs of intersecting planes.

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* This common absolute term is

$$H + (1 - \lambda^2)\, l^2 + (1 - \mu^2)\, m^2 + (1 - \nu^2)\, n^2.$$

(B) One modulus unity and other two equal; paraboloids of revolution, including circular cylinders.

(C) Two moduli unity; parabolic cylinders, including pairs of parallel planes.

(D) Three moduli unity; single planes.

The principle on which this classification is founded is, that when a modulus is unity an axis is necessarily infinite. This amounts to supposing that one of the axes of the modular surface has been taken equal to the arbitrary line K. The moduli may have all possible values, real or imaginary, with the single restriction that the three be not together imaginary, unless imaginary surfaces be included in the generation.

I. *Central Surfaces.* When the moduli differ from unity, the coefficients of the squares of the variables in equations (1), (2), and (3) do not vanish; the surfaces are therefore central. By making the assumptions

$$x_1 - \lambda^2 x_2 = 0, \quad y_1 - \mu^2 y_2 = 0, \quad z_1 - \nu^2 z_2 = 0,$$

these equations will be referred to the common centre, and become

$$(4)\ (1 - \lambda^2)\, x^2 + (1 - \mu^2)\, y^2 + (1 - \nu^2)\, z^2 = H,$$

$$(5)\ \left(\frac{1 - \lambda^2}{\lambda^2}\right) x_1^{\,2} + \left(\frac{1 - \mu^2}{\mu^2}\right) y_1^{\,2} + \left(\frac{1 - \nu^2}{\nu^2}\right) z_1^{\,2} = H,$$

$$(6)\ (1 - \lambda^2)\, \lambda^2 x_2^{\,2} + (1 - \mu^2)\, \mu^2 y_2^{\,2} + (1 - \nu^2)\, \nu^2 z_2^{\,2} = H,$$

in which it is to be observed that the absolute terms H, H_1, H_2 are now all equal to each other.

Comparing the first and second of these with the equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

$$\frac{x_1^{\,2}}{a_1^{\,2}} + \frac{y_1^{\,2}}{b_1^{\,2}} + \frac{z_1^{\,2}}{c_1^{\,2}} = 1,$$

we obtain other equations:

$$a^2 = \frac{H}{1 - \lambda^2}, \quad a_1^{\,2} = \frac{H\lambda^2}{1 - \lambda^2},$$

$$(7)\quad b^2 = \frac{H}{1 - \mu^2}, \quad b_1^{\,2} = \frac{H\mu^2}{1 - \mu^2},$$

$$c^2 = \frac{H}{1 - \nu^2}, \quad c_1^{\,2} = \frac{H\nu^2}{1 - \nu^2},$$

which give, by subtracting them in pairs, the relations

$$a^2 - a_1^2 = b^2 - b_1^2 = c^2 - c_1^2 = H,$$

whence the confocality of the focal and generated surface is clearly seen, and also the dependence of the quantity H on their axes.

Equations (7) determine the axes of the generated and focal surfaces from the original data. These were the assumed coordinates and moduli. Being given, they determine the quantity H, and thence the axes by means of these equations.

But we can proceed in an inverse order, and from being given the two confocals, determine the moduli by means of which one of these might be generated from the other. The connexion of these required moduli with the axes of the given confocals is obtained by dividing equations (7) in pairs, whence we obtain

$$\lambda^2 = \frac{a_1^2}{a^2}, \quad \mu^2 = \frac{b_1^2}{b^2}, \quad \nu^2 = \frac{c_1^2}{c^2},$$

shewing that *the moduli are the ratios of the axes of that surface which is taken for focal to the axes of the other.* We thus see that any central surface of the second order may be considered as the locus of foci (in this extended sense) of all others confocal with it, and that by a proper selection of moduli the several surfaces of the confocal system may be generated from it according to the principles already laid down.

The moduli being the ratios of the axes of the two confocal surfaces, if the focal be fixed and the several surfaces of the system be generated from it, the corresponding moduli will vary *inversely* as the axes of the generated surfaces. The focal surface in this case is analogous to a focal curve when the surfaces of the system are generated from it by means of a modulus and directrix.

We may on the other hand view a given surface as generated by means of different sets of moduli from different focal surfaces. In this case the moduli will vary as the axes of these focals. This is analogous to the two cases of generating a surface from its two real focal curves with different moduli.

When the focal and generated surfaces are given, and the moduli thence known, the dirigent surface must next be determined. This is done by means of the relation of reciprocality already shewn to exist between this and the other two surfaces; which implies that its axes are third proportionals to the

corresponding pairs of axes of the others. When the dirigent surface is known, the dirigent on it corresponding to an assumed focus is determined by the simple construction of drawing a normal to the focal surface at the focus; the point in which this normal meets the dirigent surface is the required dirigent. The proof of this depends on the two principles which have been already stated, but can now be established; viz. that the focus and dirigent are both *reciprocal* and *corresponding* points (in the common sense) on their respective surfaces. From the expressions for the moduli in terms of the axes, we can easily deduce the relations

$$\frac{x_1}{x_2} = \frac{a_1^2}{a^2}, \quad \frac{y_1}{y_2} = \frac{b_1^2}{b^2}, \quad \frac{z_1}{z_2} = \frac{c_1^2}{c^2},$$

which shew that these points are *reciprocal* points with respect to the generated surface: and since for the ratios of the squares of these axes we may substitute the simple ratios of the axes of the focal and dirigent surfaces, which give the relations

$$\frac{x_1}{x_2} = \frac{a_1}{a_2}, \quad \frac{y_1}{y_2} = \frac{b_1}{b_2}, \quad \frac{z_1}{z_2} = \frac{c_1}{c_2},$$

it appears that they are also *corresponding* points on their own surfaces respectively.

This relation of corresponding points enables us to state the general principle in a very convenient form; viz.

The ratio of the distances of any point on the surface from two corresponding points of the focal and dirigent surfaces is equal to that of the arbitrary line K to the modular radius R.

It will be unnecessary to enter into any consideration of the several kinds of central surfaces; a slight examination will be sufficient to shew that all varieties may be obtained by assuming proper values for the moduli: when two moduli are equal, the surfaces are evidently of revolution, and since these equal values may be either greater or less than unity, real or imaginary, and the third modulus also either real or imaginary, it is obvious that every variety of this class may be obtained. Cones, real and imaginary, are obtained when the original data are such as to make H vanish, whatever the moduli be. When the sum of the squares of two moduli equals *two*, a class of surfaces is obtained which are in a certain degree analogous to surfaces of revolution, namely those in which the sections perpendicular to one of its axes are *equilateral* hyperbola. The remaining axis may be either

finite or infinite, giving in the latter case equilateral hyperbolic paraboloids and cylinders.

We shall now proceed to consider non-central surfaces.

II. *Non-central surfaces.* When one or more of the moduli become equal unity, the coefficient of the square of the corresponding variable vanishes, and the surface is therefore non-central. Another reason for this appears from what has been shewn with respect to central surfaces; viz. that the modulus is equal to the ratio of the axes of the two confocal surfaces. Now that the modulus should be unity, requires that these axes be equal, which cannot in confocal surfaces happen unless they be infinite.

The equation of non-central surfaces when one modulus equals unity may be put into the form

$$(1-\mu^2)\,y^2+(1-\mu^2)\,z^2-2\,(x_1-x_2)\,x=H.$$

Denoting the quantity $(x-x_2)$ by $2P$, this equation and those of the focal and dirigent surfaces will be respectively,

$$(8)\quad (1-\mu^2)\,y^2+(1-\nu^2)\,z^2-4Px=H,$$

$$(9)\quad \left(\frac{1-\mu^2}{\mu^2}\right)y_1^{\,2}+\left(\frac{1-\nu^2}{\nu^2}\right)z_1^{\,2}-4Px_1=H-P^2,$$

$$(10)\quad (1-\mu^2)\,\mu^2y_2^{\,2}+(1-\nu^2)\,\nu^2z_2^{\,2}-4Px_2=H+P^2;$$

the forms of which shew that the three surfaces are paraboloids having a common axis. By a comparison of the first two equations with those of paraboloids referred to a point on the common axis as origin,

$$\frac{y^2}{4p}+\frac{z^2}{4q}=x+r,$$

$$\frac{y_1^{\,2}}{4p_1}+\frac{z_1^{\,2}}{4q_1}=x_1+r_1,$$

we obtain the relations

$$p-p_1=q-q_1=r-r_1=P,$$

from which the confocality of the surfaces appears, and the connexion of the quantity P with the parameters is determined.

The values of the two moduli μ and ν can easily be shewn to be

$$\mu=\sqrt{\frac{p_1}{p}},\quad \nu=\sqrt{\frac{q_1}{q}},$$

which are analogous to those of central surfaces, and may be either real or imaginary.

All the conclusions which have been proved for central surfaces will now apply here. A paraboloid whether elliptic or hyperbolic is the locus of foci of all others confocal with it. Any two paraboloids of a confocal system being assumed, whichever is made the focal surface, the dirigent surface is its polar reciprocal with respect to the other, and the dirigent and focus are connected as before by the relations of being both *reciprocal* and *corresponding* points. *The moduli are the square roots of the ratios of the parameters of the focal to those of the generated surface.*

The value of P in the above equations is arbitrary, and depends on the original data. If P vanish, the surfaces are cylindrical, and may be either elliptic or hyperbolic. When in this case H also vanishes, we obtain the cases of intersecting planes and right lines.

The remaining cases, when two or even the three moduli are equal unity, can very easily be discussed after what has been already established: it is evident that in the former case the surfaces are parabolic cylinders, and in the latter single planes. It is sufficient to indicate this, proceeding now to point out some general principles which are true for all surfaces of the second order, and shew more fully the analogy of these new foci to the foci of focal curves.

The principles alluded to are the following; they can very easily be deduced from what has been already established.

1. *The angle at focus subtended by a chord of the surface which passes through the dirigent, is bisected either internally or externally by the line joining these points.*

2. *A tangent to the surface from dirigent subtends a right angle at focus.*

3. *The cone whose vertex is at focus, and stands on a section of the surface whose plane passes through dirigent, has line joining focus and dirigent for an axis.*

4. *A chord of the dirigent surface meets the surface in two points, the sum of the distances of which from the two foci corresponding to the extremities of the chord are equal.*

5. *If a chord of a constant length move parallel to itself inside the dirigent, it will generate a cylinder of the second order; and the sum of the distances of the points in which it*

meets the surface from the foci corresponding to its extremities will be constant.

6. *The locus of points on the surface for which the ratio of focal and dirigent distance is constant, is a spherical curve whose projections on the principal planes are conic sections; whence it can also be shewn that the circular sections of the surface are parallel to those of the modular.*

The modular and umbilicar generations by means of the focal curves may now be deduced from the general method: these are the particular cases in which the focal surfaces become the flattened spaces bounded by the focal curves. Now, since in such cases one of the axes of the focal surface vanishes, the modulus corresponding also vanishes, and the axis of the modular surface becomes infinite, as also that of the dirigent surface. These are therefore cylinders whose edges are perpendicular to the plane of the focal curve: but the peculiarity of the case consists in this—that the dirigent corresponding to any point of the flattened surface not on the focal curve is at infinity, while those corresponding to points on this curve are indeterminate (since an infinity of normals can be drawn there) and infinite in number: what really happens is—that the dirigent stretches out into a line, the *directrix*.

The general principle must however still apply to this infinite number of dirigents as well as to one. It is therefore still true that the ratio of the dirigent and focal distances is represented by the modular radius. Let now from any point of the surface a line be drawn to the directrix parallel to a circular section of the modular surface; the ratio of this line to the focal distance of the point will be equal to the ratio of the radius of the circular section to the line K, and therefore constant. Hence the modular generation is included as a particular case under the general one, and the value of its modulus obtained.

This modulus is the ratio of the line K to the radius of the circular section of the modular cylinder, (its directive semiaxis): but this is one of the moduli of this surface which may be called its directive modulus; whence we come to the conclusion, that *the modulus of the modular generation from a focal curve is the directive modulus of the modular surface.*

But before we proceed, it will be necessary to examine into the position of this axis; for if it be perpendicular to the plane of the focal curve, it will be impossible to draw

lines from every point of the surface to the directrix as above required. Hence the following principle:

When the modular surface is an elliptic cylinder, the focal curve is modular; but when hyperbolic, umbilicar: the modular generation only applies in the former case, and the umbilicar only in the latter.

The reason of this is obvious; for when the cylinder is elliptic its circular sections are inclined to its edges, whereas when it is hyperbolic they are parallel to its asymptotic planes, and therefore (as in the present case) perpendicular to the plane of the focal curve. The directive axis in this case is the infinite axis; whence we come to the well-known conclusion that the focal curve in a plane perpendicular to a directive axis is non-modular, and that those in the principal planes through that axis are modular.

The inclination of the plane of the circular section to the principal plane is now obtained at once: let ϕ be this angle, its cosine is evidently the ratio of the lesser to the greater axis of the modular cylinder. Hence if λ_1 and μ_1 be the corresponding moduli, of which λ_1 is the directive modulus or modulus (in the old sense) of the focal curve, we have $\cos\phi = \frac{\lambda_1}{\mu_1}$, from which, by substituting the values for λ_1 and μ_1, viz. $\frac{a_1}{a}$ and $\frac{b_1}{b}$, we obtain the known expression for this angle, viz.

$$\cos\phi = \frac{b}{a}\sqrt{\left(\frac{a^2 - c^2}{b^2 - c^2}\right)}.$$

In the same way let λ_2 and ν_2 be the moduli of the modular cylinder corresponding to the focal curve in the other principal plane which passes through the directive axis, and ϕ' the corresponding inclination, we have also

$$\cos\phi' = \sin\phi = \frac{\lambda_2}{\nu_2} = \frac{c}{a}\sqrt{\left(\frac{a^2 - b^2}{c^2 - b^2}\right)}.$$

From these can easily be deduced the equation given by Professor MacCullagh,

$$\frac{\cos^2\phi}{\lambda_1^2} + \frac{\sin^2\phi}{\lambda_2^2} = 1;$$

which is general, and holds for all surfaces of the second order as well when one of the two moduli is imaginary as when they are both real.

By substituting for λ_1, λ_2 their values in terms of the axes, we obtain

$$\lambda_1 = \frac{\sqrt{(a^2 - c^2)}}{a}, \quad \lambda_2 = \frac{\sqrt{(a^2 - b^2)}}{a},$$

the well-known expressions for the moduli, a being here the directive axis of the surface.

We must now prove that when the modular generation fails, the umbilicar can be used. This proof depends on the property of an hyperbolic cylinder, that the rectangle under the perpendiculars dropped from a point of it on its asymptotic planes is constant. Now conceive two planes through a directrix parallel to the asymptotic planes of the modular cylinder. Let D be any point on this directrix, F the focus, S a point on the surface, and SP, SP' two perpendiculars from S on these planes. From any point O on the axis of the modular cylinder, let OA be drawn parallel to SD to meet this surface in A, and let AQ, AQ' be the two perpendiculars from A on its asymptotic planes. It can easily be seen that if the ratio of SD to SF is represented by OA, the ratio of the rectangle under SP and SP' to the square of SF, will be represented by the rectangle under AQ and AQ', which being a constant quantity, we come to the fundamental principle of the umbilicar generation, viz: The ratio of the square of the focal distance to the rectangle under the perpendiculars on two planes through the directrix parallel to the directive planes is constant.

The value of this ratio can very easily be shewn to be equal to $K^2\left(\frac{1}{A^2} + \frac{1}{B^2}\right)$ where A and $B\sqrt{-1}$ are the axes of the hyperbolic cylinder, or equal $(\lambda^2 - \mu^2)$ in terms of the corresponding moduli. If into this we introduce the axes of the generated surface, we have for this ratio

$$c^2\left(\frac{a^2 - b^2}{a^2 b^2}\right) = c^2\left(\frac{1}{b^2} - \frac{1}{a^2}\right),$$

in which it is to be observed that c is the directive axis of the surface.

Having thus shewn that the modular and umbilicar generations are contained in the general case, it will be unnecessary to enter into further details, as those particular cases have been already so fully developed by Professor Mac Cullagh in the paper referred to.

Trinity College, Dublin, March 17th, 1849.

EXERCISES IN QUATERNIONS.

By Sir William Rowan Hamilton.

1. Although the following paper, or series, will be founded on the same *principles* as the communications on Symbolical Geometry in the present *Journal*, and on Quaternions in the *Philosophical Magazine*, yet its *plan* will be in many respects different. And the writer hopes that without either, on the one hand, too much repeating from those papers, or, on the other hand, interfering with their continuation, he may be able, by the remarks, rules, formulæ, and examples which will be submitted in the present Exercises, to give some acceptable assistance to those mathematicians, or to those mathematical students, who may do him the honour of desiring to familiarise themselves with the Calculus of Quaternions; or who may even be disposed to give that Calculus a trial, as a branch of symbolical science, and as an instrument of geometrical and physical research.

2. The general *conception* of *directed lines* has occurred to several authors; and many have also perceived the existence of an important, and indeed fundamental *analogy* between the geometrical composition and decomposition of *motions*, and the algebraical addition and subtraction of positive and negative *numbers:* which analogy has been felt to be so close and strong, as to invite and justify the application of the *names* and *marks* of addition and subtraction to the operations of constructing the intermediate and transverse diagonals of a parallelogram, when two coinitial sides of that parallelogram are given, as two directed lines, of which (what thus come to be called) the *sum* and *difference* are to be taken. With these, as *preliminary conceptions*, to the introduction of which into science the present writer is aware that he cannot in any manner pretend, he wishes to be allowed to regard his readers as being *already* thoroughly familiar, *before* entering on the study of the quaternions: although that study will certainly tend to impress them still more on the mind. They are, indeed, *common* to his own and to several other systems, to which systems, in *other* respects, the theory of the quaternions offers rather a *contrast* than a resemblance.

3. Yet, at this stage, he desires to invite attention to an unusual mode of *notation*, which appears to him to embody under a convenient and simple form, and under one which adapts itself as readily to *lines in space* as to lines in a single

plane, those preliminary and fundamental conceptions. He denotes, by the symbol

$$\text{B} - \text{A} \dots\dots\dots\dots\dots\dots (1),$$

that finite straight line which is drawn *from* the point A *to* the point B; and which line, when *thus* denoted, is understood to have a determined length, direction, and situation in space, as soon as the two points A and B themselves are supposed to receive determined positions. And then, by merely writing the two formulæ,

$$(\text{C} - \text{B}) + (\text{B} - \text{A}) = \text{C} - \text{A} \dots\dots\dots (2),$$

$$(\text{C} - \text{A}) - (\text{B} - \text{A}) = \text{C} - \text{B} \dots\dots\dots (3),$$

he *expresses*, under the form of two *identities*, those conceptions of the geometrical addition and subtraction of directed lines, as analogous to the composition and decomposition of motions, and as performed according to the same rules, in which conceptions themselves he has already carefully disclaimed all private or personal property. For, with the use above explained of the notation (1), the *geometrical identity* (2) expresses that if the directed line to C from B be *geometrically added* to the directed line to B from A, according to the rules of the composition of motions, the *geometrical sum*, or the *resultant line*, will be that *third* directed line which is drawn to C from A; whatever may be the positions in space of the three points ABC. And the *other* geometrical identity (3) expresses in like manner this converse proposition, that if the directed line from A to B, which is, by the notation (1), denoted by the symbol B − A, be *geometrically subtracted* from the directed line C − A, which is drawn from A to C, according to the laws of decomposition of motion, the *geometrical difference* obtained by this subtraction will be that directed line C − B, which is drawn from the final point B of the subtrahend line B − A to the final point C of that other given and coinitial directed line C − A, from which the subtraction is to be performed.

4. Instead of compounding only *two* successive motions, or *rectilinear steps* in space, we may compound *any number* of such steps; or, in other words, instead of considering a triangle ABC, we may consider a *polygon* ABCDE. . . . : and the known results for such more complex cases may still be expressed with great simplicity, and under the form of *geometrical identities*, by adhering to the same method of notation. Thus, for a rectilinear quadrilateral, ABCD,

whether this be or be not in one plane, we shall always have the formula

$$(\text{D} - \text{C}) + (\text{C} - \text{B}) + (\text{B} - \text{A}) = \text{D} - \text{A} \ldots\ldots (4);$$

for a pentagon, whether plane or gauche,

$$(\text{E} - \text{D}) + (\text{D} - \text{C}) + (\text{C} - \text{B}) + (\text{B} - \text{A}) = \text{E} - \text{A} \ldots (5);$$

and similarly for other polygons, in space or in one plane: the line which is drawn from the initial to the final point of any *unclosed* polygon being regarded (in this as in many other systems) as the *geometrical sum of all the successive sides* of the figure which it thus serves to *close;* exactly as in the formula (2) the *directed base* C – A of the triangle ABC was the geometrical *sum* of the two successive *sides* B – A and C – B, obtained by *adding* the second of those two sides to the first.

5. If the closing line be drawn in the order of succession of the sides, or in the order of the motion along the polygon which has been above supposed to be performed; or if the polygon be given as closed; then the *sum of all* the successive lines, including the closing line, will be a *null line*, because the motion thus conceived would simply bring a moving point *back* to its original or initial position. Accordingly, in the notation above proposed, we shall have the following formulæ of identity, which *express* this conception of *return:*

$$\text{A} - \text{A} = (\text{A} - \text{B}) + (\text{B} - \text{A}) = (\text{A} - \text{C}) + (\text{C} - \text{B}) + (\text{B} - \text{A}) = \&\text{c.} \ldots (6).$$

6. We may agree to *suppress the symbol of a null line*, when it occurs as written to the *left-hand* of any complex symbol denoting the result of any geometrical addition or subtraction; and then, by changing C to B in the identity (2), and to A in the identity (3), we shall obtain the formulæ,

$$+ (\text{B} - \text{A}) = (\text{B} - \text{B}) + (\text{B} - \text{A}) = \text{B} - \text{A} \ldots\ldots (7);$$

$$- (\text{B} - \text{A}) = (\text{A} - \text{A}) - (\text{B} - \text{A}) = \text{A} - \text{B} \ldots\ldots (8);$$

which allow of our interpreting the two isolated but *affected* symbols of lines, $+ (\text{B} - \text{A})$ and $- (\text{B} - \text{A}) \ldots\ldots\ldots (9)$,

as denoting respectively the directed line B – A *itself*, and the *opposite* of that line, namely the directed line A – B: two lines being said to be mutually opposites, when the beginning and end of the first line coincide respectively with the end and the beginning of the second. A null line is *its own* opposite,

$$+ (\text{A} - \text{A}) = - (\text{A} - \text{A}) \ldots\ldots\ldots (10);$$

but any *actual line*, that is, a line B − A with any finite length, is distinguished from the opposite line A − B by the contrast between their *directions*. A line B − A may be *subtracted* from another line C − A, by *adding* the second line C − A to the *opposite* A − B of the first; for we have, by the identities (2) and (3),

$$(C - A) - (B - A) = C - B = (C - A) + (A - B) \ldots (11).$$

7. Two directed lines being regarded as *equal* to each other, when, and only when, their *directions* as well as their *lengths* are identical, although their *situations* will generally be different, the *equation*

$$D - C = B - A \ldots\ldots\ldots\ldots (12)$$

will express, generally, that the four points ABDC are the four successive corners of a *parallelogram*; the corner D being opposite to A, and C to B: and we may still retain this mode of speaking, even when the altitude or the area of the parallelogram vanishes, by the four points ABDC coming to range themselves on one right line. From this signification of the equation (12) it is evident that this equation admits of *inversion*, and of *alternation*, so that it may be written thus (inversely),

$$C - D = A - B \ldots\ldots\ldots\ldots (13),$$

because the opposites of equal lines are equal; or thus (alternately),

$$D - B = C - A \ldots\ldots\ldots\ldots (14),$$

by Euclid I. 33, or because the two *paths of transport*, from A and B to C and D respectively, must be *themselves* equal directed lines, in order to allow of the first given directed line B − A being carried, *without rotation*, by the simultaneous motion of its two extreme points along those two paths of transport, so as to come to *coincide* with the second given directed line D − C; which second line would not be (in the foregoing sense) *equal* to the first, unless this perfect coincidence could be effected by such transport without rotation. (The writer may remark, in passing, that he agrees with those who hold that all such considerations as these, of *motions abstracted from causes of motion*, do not vitiate, in any degree however small, the *purity* of geometrical science: to think otherwise would be indeed, as he conceives, to condemn, so far, those ancient geometers, including Euclid, who generated surfaces, for example the sphere, by motion.) Directed lines which are *equal* to the same directed line are also equal to each other; and the sums and differences of

equal directed lines, similarly taken, are equal directed lines. Lines which are opposites of equal directed lines may be said, by an extension of the former definition of opposites, to be themselves also opposite lines.

8. Since, under the condition expressed by the equation (12), the line D − A is the *directed diagonal* of the parallelogram ABDC, intermediate between the two directed sides B − A and C − A, and coinitial with them, it ought (by a known principle above mentioned) to be found to be their geometrical sum: and, accordingly,

$$\mathrm{D} - \mathrm{A} = (\mathrm{D} - \mathrm{C}) + (\mathrm{C} - \mathrm{A}) = (\mathrm{B} - \mathrm{A}) + (\mathrm{C} - \mathrm{A}) \ldots (15);$$

or, adding the sides in a different order, and employing the principle of *alternation*, whereby we pass from the equation (12) to (14),

$$\mathrm{D} - \mathrm{A} = (\mathrm{D} - \mathrm{B}) + (\mathrm{B} - \mathrm{A}) = (\mathrm{C} - \mathrm{A}) + (\mathrm{B} - \mathrm{A}) \ldots (16).$$

It is therefore allowed to *change the order* of the summands in the addition of any two directed lines; a conclusion which is easily extended to any number of such lines, in space or in one plane, so as to shew that geometrical *addition* is a *commutative operation*, or that the *sum* of any given system of directed lines is always equal to the same given directed line, in whatever *order* the summation is effected. Addition of directed lines is also an *associative* operation, in the sense that any number of successive summands may be collected or *associated* together (as is done in calculation by enclosing their symbols within brackets) into one partial group, and their sum then added as a single summand to the rest: for this comes merely (when its geometrical signification is examined) to drawing a *diagonal of a polygon*, plane or gauche. It is understood that in order to avail ourselves of the identity (2), for the purpose of adding an *arbitrary* but given line B′ − A′ to another given line B − A, when the beginning A′ of the proposed *addend* line does not *already* coincide with the end B of the line to which the addition is to be performed, we are to *make* it coincide, by a transport without rotation; this process of construction being symbolically expressed by the formula

$$(\mathrm{B}' - \mathrm{A}') + (\mathrm{B} - \mathrm{A}) = \mathrm{C} - \mathrm{A}, \text{ if } \mathrm{B}' - \mathrm{A}' = \mathrm{C} - \mathrm{B} \ldots (17).$$

9. When three points ABC are so related as to satisfy the equation

$$\mathrm{C} - \mathrm{B} = \mathrm{B} - \mathrm{A} \ldots\ldots\ldots\ldots\ldots (18),$$

which gives, by principles and notations already explained,

$$\mathrm{C} - \mathrm{A} = (\mathrm{B} - \mathrm{A}) + (\mathrm{B} - \mathrm{A}) \ldots\ldots\ldots (19);$$

then, by a natural and obvious use of numerical coefficients, we may write also, as other expressions for the same relation of position between the three points (namely that the point B bisects the straight line connecting A and C), the two following equations, of which each includes the other:

$$\text{C} - \text{A} = 2(\text{B} - \text{A}); \quad \text{B} - \text{A} = \tfrac{1}{2}(\text{C} - \text{A}) \ldots. (20).$$

And generally, if a denote any positive or negative number, whether integral or fractional, and whether commensurable or incommensurable, the notation

$$\text{C} - \text{A} = a(\text{B} - \text{A}) \ldots\ldots\ldots\ldots (21)$$

may conveniently be employed to express that the point C is situated on the same indefinite right line as the points A and B, being at the same side of A as B if the coefficient a be positive, but at the opposite side of A if a be negative, and at a distance from A which bears to the distance of B from A the ratio of $\pm a$ to 1. When the coefficient a becomes zero, then both members of (21) become null lines, and C coincides with A. With such an use of coefficients we shall have, as in ordinary algebra, the two identities

$$(a' \pm a)(\text{B} - \text{A}) = a'(\text{B} - \text{A}) \pm a(\text{B} - \text{A}) \ldots. (22);$$

$$a\{(\text{B}' - \text{A}') \pm (\text{B} - \text{A})\} = a(\text{B}' - \text{A}') \pm a(\text{B} - \text{A}) \ldots (23);$$

which we may express in words by saying that the operation of *multiplication* of a directed line by a numerical coefficient is a *distributive operation*, whether relatively to the multiplying number, or relatively to the multiplied line; this *distributive* property of such *multiplication* of lines by numbers depending mainly on the *commutative* property of the *addition* of lines among themselves.

10. The equation (21) expresses sufficiently that the point C is situated *somewhere* upon the indefinite straight line which passes through the two points A and B, or that the *three* points A B C are *collinear;* and it expresses *nothing more* than this relation of collinearity, if we conceive the number a to remain undetermined. The formula (21) may therefore, with this use of an indeterminate numerical coefficient a, be regarded as the *equation of an indefinite right line in space; one such equation* being *sufficient* in this mode of dealing with the subject. This equation (21) may however be made to assume a more symmetric form, by introducing the consideration of an arbitrary fourth point D, supposed to be situated anywhere in space, with which the three collinear

points ABC shall be compared. For thus, by writing the equation successively under the following forms,

$$(\text{C} - \text{D}) - (\text{A} - \text{D}) = a(\text{B} - \text{D}) - a(\text{A} - \text{D}) \ldots\ldots (24),$$

$$\text{C} - \text{D} = a(\text{B} - \text{D}) + (1 - a)(\text{A} - \text{D}) \ldots\ldots (25),$$

$$(b - ba)(\text{A} - \text{D}) + ba(\text{B} - \text{D}) - b(\text{C} - \text{D}) = 0 \ldots (26),$$

$$l(\text{A} - \text{D}) + m(\text{B} - \text{D}) + n(\text{C} - \text{D}) = 0 \ldots\ldots (27),$$

in each of the two last of which forms the symbol for zero is understood to denote a null line, we see, by comparing these two forms, that the coefficients l, m, n, in the form (27), are connected among themselves by the equation of condition

$$l + m + n = 0 \ldots\ldots\ldots\ldots (28);$$

and, conversely, that under this last condition the formula (27) expresses that the three points ABC are ranged upon one common right line. In fact, when the condition (28) is satisfied, we can eliminate the coefficient l by it from (27), and so obtain the equation

$$m(\text{B} - \text{A}) + n(\text{C} - \text{A}) = 0 \ldots\ldots\ldots (29),$$

which evidently agrees with the form (21), and conducts to similar consequences.

11. Let E be a new point situated anywhere upon the indefinite straight line CD; and therefore satisfying an equation of the form

$$p(\text{E} - \text{D}) = n(\text{C} - \text{D}) \ldots\ldots\ldots\ldots (30),$$

where n may denote the same coefficient as in (27). Eliminating this coefficient n, the symbol C for the point of intersection of the two indefinite straight lines AB, DE, disappears; and there results, as the expression of the condition that *some* such point C shall exist, or that the *four points* ABDE shall be situated in *one common plane*, an equation of the following form,

$$l(\text{A} - \text{D}) + m(\text{B} - \text{D}) + p(\text{E} - \text{D}) = 0 \ldots\ldots (31);$$

which seems to resemble the equation (27), but differs from it in this important respect, that the sum of the three new coefficients lmp does not now generally vanish, as the sum of the three old coefficients lmn did vanish, in virtue of the condition (28). And by comparing the *equation of coplanarity* (31) with the *condition of collinearity* (28), we may now see that this last-mentioned condition (28), in combination with the equation (27), expressed that the three given

points ABC are *coplanar with any arbitrary fourth point* D; which can only be by those *three* points ABC being *collinear* with each other. In fact, under the condition (28), we have

$$l(\mathrm{D} - \mathrm{D}') + m(\mathrm{D} - \mathrm{D}') + n(\mathrm{D} - \mathrm{D}') = 0 \ldots . (32),$$

by adding which to (27) we obtain the same result, namely

$$l(\mathrm{A} - \mathrm{D}') + m(\mathrm{B} - \mathrm{D}') + n(\mathrm{C} - \mathrm{D}') = 0 \ldots . (33),$$

as if we had simply changed the symbol of the fourth point D to that of any arbitrary fifth point D′. By introducing the symbol O of a new and arbitrary point of space, with which the four coplanar points ABDE may be compared, through drawing lines from it to them, the equation of coplanarity (31) assumes the form

$$l(\mathrm{A} - \mathrm{O}) + m(\mathrm{B} - \mathrm{O}) + n(\mathrm{D} - \mathrm{O}) + p(\mathrm{E} - \mathrm{O}) = 0 \ldots (34);$$

where n is a new coefficient, connected with the others by the condition

$$l + m + n + p = 0 \ldots\ldots\ldots\ldots \quad (35).$$

These remarks, and a few others which shall be offered in some following articles, may be of use, as serving to illustrate and exemplify an unusual mode of *notation** in geometry; but they can only be regarded as *preparatory* to the theory of the *quaternions*, because that theory, in its *geometrical* aspect, depends essentially on the conception of the *multiplication* and *division* of *one* directed line *by another line* of the same kind, and not merely by a numerical *coefficient*: a QUATERNION (in the author's system) being always equal to such a *product or quotient of two directed lines in space.*

[*To be continued.*]

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* The notation B − A, for a *directed right line in space*, was proposed in a note to the first article of the paper on Symbolical Geometry, printed in the present *Journal* (towards the end of 1845): and it had been long familiar to the writer, as an extension to *space* of a similar notation relatively to *time*, which had been published by him in the year 1835, to express a *time-step*, or directed interval in time, from any *one moment* (not number) denoted by A, to any *other moment* of time denoted by B. (See the Essay on Algebra as the Science of Pure Time, in the 1st part of the XVIIth volume of the *Transactions of the Royal Irish Academy*). With respect to the mere fact of *distinguishing* between the two elementary geometrical symbols, AB and BA, as denoting two *opposite lines*, the present author cheerfully acknowledges that this simple and natural distinction has often been noticed and employed by other writers on Geometry.

ON THE EQUILIBRIUM OF A FLOATING BODY.

By Richard Townsend.

Having been led to consider the problem of the equilibrium of a floating body from an observation of Poisson (*Traité de Mécanique*, Art. 606), that it would be difficult to prove, *a priori*, geometrically, that for every floating body, whatever be its external form or internal structure, there must exist a position of equilibrium; the following solution presented itself, which is possibly as simple and at the same time as general as could perhaps be desired. It depends on the following principle.

The problem of the equilibrium of every floating body may be always reduced to that of another body standing on a horizontal plane; the auxiliary body being bounded by a surface determinable within the original body, and the two bodies (both of course of finite dimensions) having the same centre of gravity.

To prove which we shall shew that

In every floating body, whatever be its structure or form, there may be traced out a certain determinable surface such that if the body be placed on the fluid displacing the proper volume due to its mass in any position, whether of equilibrium or not, the upward vertical force resultant of the pressures of the fluid will be always normal to that surface.

And the surface possessing that property within the body, supposed placed successively in every position displacing the same quantity of fluid of mass equal to its own, is the locus of the centres of gravity of all the displaced volumes.

Admitting this for a moment, and therefore with it the principle evident from it, the following consequences are at once manifest.

Every floating body must have at least two positions of equilibrium, one stable and the other unstable; for, as in the case of a body standing on the ground, of all the radii drawn from its centre of gravity to the auxiliary surface within it, one must be an absolute maximum and another an absolute minimum, and the body being placed with either of these vertical will, in the former case, be in a position of unstable and in the latter of stable equilibrium, inasmuch as these lines are both normals to the auxiliary surface, and the two centres of curvature of that surface corresponding to the point of meeting are in the former case both vertically below and in the latter both vertically above the centre of gravity of the body.

For every floating body there exists in general a determinate number of positions of equilibrium, stable, unstable, or intermediate; for, from every arbitrary point there could in general be drawn to any surface a determinate number of normals, real or imaginary, and of these, as in the case of a body standing on the ground, every real normal from the centre of gravity of the body to the auxiliary surface would of course be a position of equilibrium. If the surface of the body were *imperfect*, as for instance in the case of a segment of a body cut off by a plane, the number of possible positions would of course be *diminished** by the number of real normals which could be drawn from the centre of gravity of the body to the absent portion of the auxiliary surface.

For every position of equilibrium of a floating body there exist in general two different metacentres; the two vertical planes of displacement corresponding to which are at right angles to each other: for those two points in any position of equilibrium are obviously the two centres of curvature corresponding to the lowest point of the auxiliary surface, and, as in the case of a body standing on the ground, they are in general at two different points on the vertical normal passing through the centre of gravity of the body: when this is the case there are obviously but *two* directions of displacement which have a metacentre at all, and those are of course at right angles to each other; but when, on the contrary, in any particular position of equilibrium the two centres of curvature coincide with each other, then will *every* direction of displacement have indifferently a metacentre, which will be always at the same elevation or depression above or below the centre of gravity of the body.

In any position of equilibrium of a floating body, in order that the equilibrium should be perfectly stable or perfectly unstable, two conditions are necessary; viz. that the two metacentres corresponding to that position be either both

* From this, however, it must not be inferred that for every such normal a position of equilibrium is actually *lost* and *ceases to exist* for the body, but merely *that it ceases to be determined by the preceding construction*, which fails *in a particular sense* in consequence of the imperfection of the bounding surface of the body entailing itself to an extent more than proportional to its amount on the auxiliary surface, and from the normals, therefore, having to be drawn from the centre of gravity of the body not to the latter surface, properly speaking, but to a portion of a different surface whose shape may follow no determinable law. This remark is to be considered as applying to every result of the present paper, as otherwise some of the conclusions arrived at would require to be suitably modified in order to be strictly accurate.

vertically above or both vertically below the centre of gravity. These, as in the case of a body standing on the ground, will obviously be expressed in algebraical language by stating in terms of given or known quantities that the two principal radii of curvature at the lowest point of the auxiliary surface in the position of equilibrium are in the former case both greater and in the latter both less than the distance of the same point from the centre of gravity of the body.

These consequences being manifest, we proceed now to the geometrical principles on which they depend.

Let XYZ represent respectively any closed surface, the surface envelope of a series of planes cutting from it a constant volume, and the surface locus of the centres of gravity of all the separated volumes. The surface X, in what follows, may be considered as that of the body, of which we may denote by M the mass, and by G the centre of gravity; the envelope Y will then be that due to the volume of the equivalent mass of the fluid, of which we may denote by V the constant volume, and by ρ the density, so that $M = V\rho$; and the third or auxiliary surface Z will be fully determined from X and Y. Let also ABC represent, respectively, the section of X made indifferently by any one of the cutting planes, and the two points of Y and Z which correspond to that plane. Then have we in general the following properties, which have manifest reference to the subject before us.

1. *The point of contact B is always the centre of gravity of the area A.*

2. *The tangent planes to Y and Z at every pair of corresponding points B and C are always parallel to each other.*

3. *The line joining any point C on Z with a consecutive point C' is always perpendicular to the line of intersection of the two consecutive tangent planes to Y at the corresponding points BB', when C' is on either of the two lines of curvature of Z passing through C; and, conversely, when the two aforesaid lines are at right angles to each other, then is C' always on one or other of the two lines of curvature of Z passing through C.*

4. *The normal and the two principal tangents at any point C of Z are always parallel to the three principal axes of the area A at its centre of gravity B.*

5. *The two principal radii of curvature at any point C of Z are always equal to the two principal moments of inertia of the*

area A round the two axes in its plane at its centre of gravity B, divided by the constant volume V.

Of these properties, the 2[nd] proves the theorem which furnishes all the positions of equilibrium, the 5[th] gives the conditions of stability or instability, and the remaining three are necessary in the proof of the last. Three surfaces connected with each other, as XYZ, possess many other highly interesting properties admitting of remarkably simple solutions, but we have here confined our attention exclusively to the few which have direct reference to our immediate subject. The order in which they have been enumerated above is that in which they may be most conveniently proved, and in which accordingly we proceed to establish them.

Denoting by BB' any two consecutive points on the surface Y, by VV' the two equal volumes cut off from X by the tangent planes at those points, by ww' the volumes of the two equal infinitely thin wedges included between those planes, by $\delta\theta$ the common angle of those wedges, by CC' the two centres of gravity of VV', by oo' those of ww', by O that of the portion of volume common to V and V', by P the line of intersection of the two tangent planes at B and B', by Q the line joining o and o', by i the angle of intersection of P and Q, by rr' the two distances Oo Oo', by ss' the two segments into which the line oo' is cut by P; and in the plane of the area A, taking for axes of x and y the right lines P and Q respectively, we proceed to prove them in order.

Property 1 has been long known and is indeed evident; w and w' being equal in volume, we have $\delta\theta . \sin \iota . \Sigma y\, dx\, dy$ for $w = \delta\theta . \sin \iota . \Sigma y' dx' dy'$ for w', hence in the area A $\Sigma y\, dx\, dy$ at one side of the axis of $x = \Sigma y' dx' dy'$ at the other side; therefore the centre of gravity of A lies in the right line P, and the same being true for the intersections of *every* pair of consecutive tangent planes, it follows that the point B is that centre of gravity.

To prove 2. We have the proportions $OC : Oo :: w : V$, and $OC' : Oo' :: w' : V'$, from which, since $V = V'$ and $w = w'$, we have the proportion $OC : OC' :: Oo : Oo'$; therefore, whatever be the position of the two tangent planes, the infinitely small right line CC' is always parallel to oo', that is ultimately to the tangent plane at B. Hence we see that a point, in moving from any point C on the surface Z to any infinitely near point on the surface, *must move always in a direction parallel to the tangent plane at B, the point corresponding to C*

on the surface Y, but it can also move only in the tangent plane to its own surface at C, therefore the two tangent planes at B and C are always parallel to each other.*

To prove 3. It is evident that if from any two arbitrary points perpendiculars be let fall upon any two arbitrary planes, they will intersect each other if the right line joining the points be perpendicular to the intersection of the planes, but otherwise they will not; and, conversely, if the perpendiculars intersect, the line joining the points will be perpendicular to the intersection of the planes, but otherwise it will not; and these being true in general are therefore true when the points are infinitely near to each other, as at C and C', and when the planes intersect at an infinitely small angle, as at B and B'.

To prove 4. The right line CC' being (by property 2) always parallel to oo', and in the particular case of a principal tangent at C being (by property 3) perpendicular to the axis of x, it follows that the two tangents at C to the surface Z drawn parallel to the axes of x and y, will be the two principal tangents at that point *when those axes are at right angles to each other*, that is, when the right line joining the centres of gravity oo' of the two equal wedges ww' is perpendicular to their common edge P; but in that case those two right lines will be the two principal axes of the area A which lie in its plane at its centre of gravity B; for, drawing through the right line oo' a plane perpendicular to the axis of x, the distance of the centre of gravity of the whole double wedge from that plane must be nothing, but the numerator of the fraction which expresses the value of that distance is $\delta\theta . \Sigma xy\, dx\, dy$, therefore the two rectangular axes of x and y are such that for them the sum $\Sigma xy\, dx\, dy = 0$, and therefore they are both principal axes of the area A at its centre of gravity B.

To prove 5. The angle between any two consecutive normals at C and C' to the surface Z being (by property 2)

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* Otherwise thus on *mechanical* principles. Considering X as the interior surface of a vessel of any shape into which some liquid has been poured occupying a volume equal to V, and moving the vessel about slowly in every direction so that the liquid may be considered as in equilibrium in every position; the constantly horizontal surface of the liquid will then form within the vessel the surface Y, and its centre of gravity will trace out the surface Z: but the liquid throughout the whole motion constituting in every position a connected system of bodies in constrained equilibrium under the action of gravity, its centre of gravity during every possible small change of position can only perpetually move horizontally, and therefore the tangent plane at any point of Z will be always parallel to the tangent plane at the corresponding point of Y.

always equal to the angle of the corresponding wedges $\delta\theta$; when those normals intersect, denoting by R the radius of curvature, we have $R.\delta\theta = CC'$, but $CC' : oo' :: OC : Oo$, or $:: OC' : Oo'$, that is $:: w : V$, or $:: w' : V'$, therefore $CC' = \frac{w}{V}.oo' = \frac{w}{V}(s + s')$; but since (by property 3) the axes of x and y are at right angles to each other, we have $s.w = \delta\theta.\Sigma y^2 dx\,dy$, and $s'.w' = \delta\theta.\Sigma y'^2 dx' dy'$, therefore as $w = w'$, we have $w(s + s') = \delta\theta.(\Sigma y^2 dx\,dy + \Sigma y'^2 dx' dy') = \delta\theta$ (the moment of inertia of the area A round the axis of x), and therefore the radius of curvature $R =$ that moment of inertia (which, by property 4, is a principal moment) divided by V.

Having thus established these properties, we proceed now to apply them to the question before us.

The auxiliary surfaces Y and Z being conceived traced out in the interior of the given body, both determinable within it from its known external surface X, its known mass M, and the known density ρ of the fluid on which it is supposed to lie. The latter of them Z is that surface which will possess the property mentioned in page 169, that if the body be supposed placed on the fluid displacing the same constant volume due to its mass in every position, whether of equilibrium or not, *to it the upward vertical force resultant of the pressures of the fluid will be always normal.* For that force in every position of the body being always vertical, is therefore always perpendicular to the plane of floatation, that is to the tangent plane to the surface Y corresponding to the position of the body; and passing always through the centre of gravity of the displaced volume of fluid, it therefore always pierces the surface Z at the point of that surface corresponding to the same position; but (by property 2) the tangent planes at every pair of corresponding points of Y and Z are always parallel to each other, therefore the resultant force of the fluid always pierces Z vertically at a point where the tangent plane is parallel to the plane of floatation, that is at a point where the tangent plane is horizontal: and therefore the force in every position of the body is always normal to that surface.

Hence (which is the principle we stated at the commencement, page 169) *the positions of equilibrium of X placed floating on the fluid, must, if they exist at all, coincide with those of Z placed standing on a horizontal plane;* and that such a position should exist, as it is only necessary that a normal to Z should pass through the point G, it follows *that there*

are always at least two positions of equilibrium, and in general a determinate number, equal to that of the real normals which could be drawn from the centre of gravity G to the auxiliary surface Z, and determined by placing the body so that any one of those normals should be in a vertical position.

In any of those positions to determine *the nature of the equilibrium.* The body in the usual manner must be supposed to receive an infinitely small displacement of the most general possible nature, and it must be ascertained whether the tendency of the different forces in the displaced position be such as necessarily or accidentally to conspire with or oppose the displacement, in whatever manner it may be arbitrarily impressed. Now the most general displacement which a solid could receive being in all cases resolveable into three rotations round any three arbitrary axes, and into these translations parallel to the same axes, in the instance before us for convenience we may take, *First*, our three axes rectangular, one vertical and the other two horizontal, which will afford the advantage that three of the six independent displacements, viz. the vertical rotation and the two horizontal translations, will manifestly, from the nature of the question, produce no effect whatever on the equilibrium of the body, and may therefore be disregarded altogether; we may take, *Secondly*, for our vertical axis, the particular vertical passing through the point of contact of the surface *Y* with the plane of floatation, which will afford the advantage that wherever the origin be assumed on that vertical, two of the three remaining displacements, viz. the two horizontal rotations, will take place without changing the volume of the fluid displaced by the body, or, at most, without altering it more than by an infinitely small quantity of an order inferior to the other small quantities under consideration, and which may therefore be neglected in comparison with them; and while therefore to such a displacement we may apply the preceding principles, we shall also, in estimating the effect of the forces, have them all constituting only a couple. And we may take, *Thirdly*, our two horizontal axes parallel to the two principal tangents to the surface *Z* at the centre of gravity of the fluid displaced in the original position of the body, which will afford the advantage that the directions of the two vertical resultants of the fluid pressures in the displaced positions of the body due to the rotations round these axes separately, will both intersect the original direction within the body of the same force in the position of equilibrium, and therefore that the moments of those two forces round the

centre of gravity of the body will be constantly in the same two fixed planes in the body. Hence therefore, without prejudice to the most extreme generality, or to its perfectly arbitrary nature, in estimating the effect of the most general displacement, angular and linear, which could be impressed on the body, we need take into account, so far as the present question is concerned: *First*, in the former case, only *a horizontal rotation* round an arbitrary axis, resolveable always into two of arbitrary magnitude in the two preceding directions, and taking place always without producing any change in the volume of the displaced fluid, and therefore *not* without producing an elevation or depression of the centre of gravity of the body proportional to the distance of that point from the vertical passing through the centre of gravity of the plane of floatation, a distance which in some particular cases may certainly be nothing, but which in general is of finite magnitude, and which on the contrary in other particular cases may be often very considerable. And *Secondly*, in the latter case, only *a vertical elevation or depression* common to all the points of the body, distinct therefore from that due to the horizontal rotation, and changing the volume of the displaced fluid by an arbitrary quantity proportional to the magnitude of the displacement multiplied by the area of the plane of floatation. Let us now consider separately the effects of these two different species of displacement.

So far as the latter alone is concerned, as the whole system of forces acting on the body constitute only a couple, tending therefore to produce only a motion of rotation round the centre of gravity, it is manifest from the preceding that the equilibrium will be stable, unstable, or intermediate, *according as the two centres of curvature at the lowest point C of the surface Z, that is the two metacentres of the floating body corresponding to that particular position of equilibrium, are both above both below, or one above, and one below the centre of gravity G*; or, in other words, *according as the two principal radii of curvature of that surface at C are both greater, both less, or one greater and one less than the distance CG.*

The expression of this last, in algebraical language, gives at once the known *conditions of stability or instability* in terms of the usual given or at least determinable quantities; for, denoting by a the distance CG, by R' and R'' the two radii of curvature, by V, as before, the constant volume of the fluid displaced by the body, and by I' and I'' the two principal moments of inertia of the area of the plane of floatation round its centre of gravity B, we have for R'

and R'' respectively (see Property 5, page 171), the values $\frac{I'}{V}$ and $\frac{I''}{V}$, and therefore for the conditions of stability or instability $\frac{I'}{V} \gtrless a$ and $\frac{I''}{V} \gtrless a$, or $I' - Va >< 0$, and $I'' - Va >< 0$. The wellknown conditions obtained first by Poisson from the equation of *vis viva* applied to the motion of the body displaced slightly from its position of equilibrium and abandoned to the action of gravity and of the surrounding fluid.—*Traité de Mécanique*, Art. 616.*

Nor will *the vertical displacement*, which we next proceed to consider, produce any effect on the stability of the equilibrium, the above conditions being fulfilled; for, whether it be impressed by itself singly or in conjunction with the horizontal displacement, it is manifest that its effect on the equilibrium *vertically* can in any case be only to produce a motion of vertical *oscillation;* and it can be easily shewn that its *mean* effect on the equilibrium *horizontally* will in all cases be nothing, or at most inconsiderable, though in all cases but one it will certainly affect very considerably the nature and circumstances of the horizontal *motion* of the body, either as producing oscillations in that direction by its own action alone, or as modifying the oscillations produced by an impressed motion of horizontal rotation.

The only forces acting on the body in any position of merely horizontal displacement being a couple, the immediate effect of the vertical displacement will be in general to alter at once both the magnitude of one of the two forces of the couple, and at the same time its distance from the other force; that is, in other words, to introduce a single force into the system, and at the same time to alter the moment of the previously existing couple. The introduced *force* being *of the same order of magnitude as the impressed displacement,* and therefore of the same order as all the other small quantities under consideration, must certainly be taken into account in estimating *the resulting motion of translation,* and it is to it in fact that the vertical oscillations are mainly due; but it may be altogether disregarded in estimating *the motion of rotation round the centre of gravity,* for, *its moment* towards producing or affecting that rotation being itself multiplied by the infinitely

* The same conditions investigated in a manner wholly different both from Poisson's and from the above have been also given by M. Moseley (see his *Hydrostatics*, Arts. 82, 83, 84), a work which I regret much not having met with until after the present paper had been forwarded for insertion to the Editor of this *Journal*.

small perpendicular from that point upon its line of direction will be an infinitely small quantity of the second order, and may therefore be neglected. Such however *is not the case* with regard to *the change of position* of the line of force; *that* must in general be taken into account, and it is to it that *the modification* of the horizontal motion of rotation *produced by a vertical displacement* is entirely due.

There is one and but one case in which the effect of this change of position in either producing or modifying the horizontal oscillations is nothing, or at least negligible—and that is, *when the centre of gravity B of the area of the plane of floatation in the position of equilibrium is in or nearly in the vertical GC, containing those of the body and of the displaced fluid;* for in that case the line $C'B'$ in the displaced position due to the horizontal rotation can make but an infinitely small angle with the vertical at C', and the effect of the vertical displacement being obviously to elevate or depress the centre of gravity C' along the line $C'B'$ by a small quantity $C'C''$ of the same order of magnitude as that of the displacement, the distance between the two verticals at C' and C'' can therefore be only an infinitely small quantity of the second order; but it is precisely that distance multiplied by the finite vertical force of the fluid which is the change in the original moment produced by the vertical displacement; the effect therefore of that introduced moment in modifying the horizontal oscillation may be regarded as insensible.

But in every other case, when the line CB makes an angle of any finite magnitude with the vertical CG, the tendency of the vertical displacement being to change the distance of the normal at C' from the centre of gravity G by a quantity of the same order of magnitude as the displacement, and therefore as that of the distance itself, its effect on the horizontal vibrations will be very considerable indeed, *causing the two metacentres in place of being both fixed to be moveable points* oscillating rapidly on either side of their original positions through certainly *finite* and possibly *very considerable* amplitudes directly proportional to the amount of displacement, and perhaps also *bringing them sometimes one or both actually below the centre of gravity G*: but notwithstanding this, though its *momentary* and unceasing effects are thus manifestly very considerable on the horizontal *motion* of the body, yet its *mean* or total effect *is nothing as regards the stability of its equilibrium;* for, as will be easily seen, throughout the whole motion the *mean positions* of the two metacentres, on which alone the nature of the equilibrium

entirely depends, ***will both remain nearly unaltered,*** inasmuch as for any given angle and direction of horizontal displacement the elevations and depressions of the body will change the positions of those points ***in opposite directions and through equal amplitudes for equal displacements,*** the elevations lowering the metacentres for one system of angles of horizontal displacement and raising them for the remaining system, and the depressions on the contrary raising them for the former system of angles and lowering them for the latter. All of which will be manifest by taking arbitrarily any centre of gravity C' on the surface Z, and by considering that the effect of any given vertical depression will be to raise that point along the line $C'B'$ by a quantity $C'C''$, directly proportional to the area of the plane of floatation, and therefore according to its position to increase or diminish its distance from the vertical CG, while on the contrary the effect of an equal elevation will be to produce exactly opposite results, precisely the same in magnitude and direction, and differing only in sign.

As the angular oscillations of a floating body round its centre of gravity are always, as we have seen, independent of the linear in the particular case of the centres of gravity of the body and of the area of the plane of floatation in the position of equilibrium, being in or nearly in the same vertical, while in every other case they are always considerably disturbed by their action; ***so reciprocally will the linear in the same case be always independent of the angular oscillations, while in every other case they will always be considerably affected by them.*** For when the line joining the two centres of gravity is normal to the surface of the fluid, then manifestly will every small angular displacement round *any* axis passing through the centre of gravity of the body always take place without altering the volume of the displaced fluid, consequently without altering the existing difference, whatever be its amount, between the upward force of the fluid and the downward weight of the body, and therefore without either exciting or modifying the body's vertical motion which is entirely due to that difference; while, on the contrary, when the line joining the two centres of gravity makes with the vertical an angle of any finite magnitude, then only for those particular axes which are in or nearly in the plane passing through that line normal to the surface of the fluid can the smallest angular rotation take place without altering the displaced fluid volume, and therefore without exciting or modifying the vertical motion

of the body. In order therefore *that the two different species of oscillations* of a floating body displaced slightly from a position of stable equilibrium and abandoned to the action of gravity and of the surrounding fluid *should be independent of each other,* it is necessary and it is sufficient *that the line joining the centres of gravity of the body and of the area of the plane of floatation should be in or nearly in the same vertical.* All this is, of course, only *on the principles of the common theory,* in which the friction, inertia, resistance, and motion of the fluid itself are entirely neglected, and in which the body—forces being supposed introduced equivalent to those of the fluid in every position—is considered to be entirely free.

In general, on the principles of that theory, the only system in motion being the body itself, and the only forces acting on that system being vertical, it follows at once from two general principles of motion, whatever be the nature or form of the body, that its motion of translation at its centre of gravity can be only vertical, and that its motion of rotation round that point must be always such that the axis of principal moment corresponding to the axis of instantaneous rotation shall be constantly horizontal,—that is, provided of course the body, when abandoned to the action of gravity and of the surrounding fluid, have received no impressed movement either of vertical rotation or of horizontal translation: hence, in no case can the body perform more than *one* system of *linear* oscillations, and that always vertical; but coexisting with that it may in general perform *three* distinct systems of *angular* oscillations round its centre of gravity, two horizontal and the third vertical, the latter being generally small, but not infinitely small as ~~compound~~ compared with the two former; and of these four systems of oscillations the angular round the centre of gravity will always, from a third general principle of motion, be independent of the linear, provided that neither species of oscillations have any effect on the forces producing the other: such in general would be the nature of the motion whatever were the structure or shape of the body, and its determination* would

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* When looking over *Moseley's Hydrostatics* on the occasion referred to in the note on p. 177, I found that this problem had been already solved by him for the most general case, for a floating body of any form displaced slightly from any position of stable equilibrium and abandoned to the action of gravity and of the surrounding fluid. See Art. 88 of his work, where the equations of the motion of the body are investigated in the most general manner and exhibited in a very elegant and symmetrical form. His management of this part of his subject appears to me far more satisfactory than that of the part already referred to.

be by a particular application of the general equations and laws of small oscillations, in which in most cases the determination of the constants is more or less tedious. But there is a particular and very extensive class of floating bodies, for which one of the four systems of oscillations is always nothing, or at least insensible, and for which at the same time the other three are all independent of each other: in this particular case, which is that of most practical importance, the motion may be obtained in a far easier manner by simply determining on the preceding principles the motions due to each of the coexisting oscillations separately, and then superposing the three motions at every point of the body.

The class of bodies for which the motion may be thus simply determined is that *for which the three principal axes at the centre of gravity in a position of stable equilibrium are parallel or nearly parallel to those of the area of the plane of floatation at its centre of gravity, and for which at the same time those two centres are in or nearly in the same vertical:* such being the conditions necessary and sufficient, the latter in order that, whatever be the displacement, the two species of oscillation, the linear and the angular, should be independent of each other, and the former in order, 1stly, that each horizontal component of the angular oscillations should be capable, if singly produced, of existing separately without exciting the other, and 2ndly that, whatever be the displacement, the vertical component should be constantly nothing or insensible, and therefore that the two horizontal components should be further capable of coexisting without disturbing each other. That the vertical component of the angular motion will in this case be always insensible, is manifest from observing that the plane of the two horizontal principal axes in the position of equilibrium remaining throughout the motion nearly horizontal, the axis of principal moment must therefore be constantly very nearly in that plane, and therefore, the plane being principal, so must also the axis of instantaneous rotation. The case of the bodies we are now considering, though apparently a little more general, as including that of every body symmetrical on each side of the two rectangular planes and floating in equilibrium with the line of their intersection vertical, is virtually the same as that discussed by Poisson (*Traité de Mécanique*, Art. 617); but *the equations of the three systems of oscillations* with *the three periods of vibration which correspond to them separately* may be obtained on the preceding principles in a manner much simpler than that by which he has obtained them.

Denoting by ζ the vertical elevation or depression of the centre of gravity G, by θ' and θ'' the two angles of horizontal rotation, by k' and k'' the two corresponding radii of gyration of the body; and, as before, by A the area of the plane of floatation, and by I' and I'' its two central moments of inertia parallel to k' and k''; then (by Property 5, p. 171, and p. 176) will the elevations of the two metacentres above the centre of gravity G be given by the expressions

$$\frac{I'}{V} - a \text{ and } \frac{I''}{V} - a,$$

or, which is the same thing, by

$$\frac{I' - Va}{V} \text{ and } \frac{I'' - Va}{V};$$

and we shall have, 1stly, for the motive force producing the vertical oscillations, the magnitude $A\rho g.\zeta$, and therefore for the equation of the vertical motion, dividing that force by the mass of the body,

$$\frac{d^2\zeta}{dt^2} + \left(\frac{A}{V}\right) g.\zeta = 0,$$

M being $= V\rho$; and 2ndly, for the motive forces producing the two systems of horizontal oscillations, the common magnitude Mg, and for the perpendiculars from the centre of gravity G upon their lines of direction which pass respectively through the two metacentres, the expressions

$$\left(\frac{I' - Va}{V}\right)\theta' \text{ and } \left(\frac{I'' - Va}{V}\right)\theta'',$$

and therefore for the two equations of the horizontal motion, dividing the moments of the two forces by the corresponding moments of inertia of the body round its centre of gravity,

$$\frac{d^2\theta'}{dt^2} + \left(\frac{I' - Va}{Vk'^2}\right) g.\theta' = 0, \text{ and } \frac{d^2\theta''}{dt^2} + \left(\frac{I'' - Va}{Vk''^2}\right) g.\theta'' = 0.$$

And from these three equations we have at once for the three periods of vibration the values, for the vertical

$$\frac{2\pi}{\sqrt{g}} \cdot \sqrt{\frac{V}{A}},$$

and for the two horizontal

$$\frac{2\pi}{\sqrt{g}} \cdot \frac{k'\sqrt{V}}{\sqrt{(I' - Va)}} \text{ and } \frac{2\pi}{\sqrt{g}} \cdot \frac{k''\sqrt{V}}{\sqrt{(I'' - Va)}}$$

respectively, values which in general are not commensurable with each other.

The preceding principles apply also to the case of *a body subject while floating in equilibrium to the incessant action of small disturbing forces arising from any cause and varying according to any law, which forces, though not sufficient to destroy the equilibrium and overturn the body, are yet such as to be capable of producing and keeping up a continued and unceasing motion of variable oscillation horizontal and vertical;* and they readily shew us, that though for the mere stability of the equilibrium it is only necessary and it is sufficient that the centre of gravity should be sufficiently low—a condition which may in actual practice be always secured by the body being sufficiently loaded or ballasted underneath—yet, what may be of quite as much practical importance, in order that the resulting motion should be as easy and regular and as little violent and abrupt as possible, that other circumstances must also be taken into account. To examine these briefly, let the whole system of disturbing forces be transferred parallel to themselves to the centre of gravity of the body, and let them then with all the generated moments be resolved each into their three components along the axes which we have been considering; we shall then have one vertical and two horizontal *forces*, which we may denote respectively by R, P, and Q; and one horizontal and two vertical *moments*, which we may also denote by L, H, and K, respectively; and of the whole six we may be supposed to know, at least approximately, the mean or average values. With three of these, as before, we have nothing to do, viz. with P, Q, and L, for there is nothing in the nature either of the body itself or of the principal force acting upon it to prevent their full effects, and the body, according to the use intended to be made of it, may either be left free to obey the resulting motions, or it may be moored or anchored so as either to modify them more or less in any manner required, or, if necessary, to prevent them altogether. It is only the other three, H, K, and R, with the motions of oscillation resulting from their action, that we have now to consider; and here again, as in a former case, we may observe that, having thus left out of consideration the only forces capable of making the body move horizontally, and there being nothing in the nature either of those retained or of those subsequently arising from gravity or from the variable action of the fluid which could impress upon it any tendency to move in that direction, *the centre of gravity can only oscillate vertically under the*

combined action of all the causes which we are actually considering. All this is of course irrespective of any horizontal motion which the body may have from other causes, or of any motion of any sort in the fluid itself, of which whenever it exists, whatever be its nature, the body from its connexion with the fluid must necessarily partake.

First then, prior to all consideration of the effects of these disturbing causes, in order that the oscillating motion of the body once excited should, whatever be its actual amount, be as regular and as little liable to abrupt changes as possible, it is manifest, in the first place, that the different *species* of oscillation of which it is susceptible, and of which the *periods* are in general incommensurable with each other, *should irrespectively of their exciting causes be all as independent of each other and of each other's variations as possible;* and this, as we have seen (p. 181), is a circumstance *which will be secured by the centres of gravity of the body and of the area of its plane of floatation in the position of equilibrium being as nearly as possible in the same vertical, and by the three central principal axes of both being at the same time as nearly as possible parallel to each other,* and which being secured will render the determination of the other requisite circumstances simpler than for a body not possessing those properties.

Next, the *periods* (like all other cases of small oscillations) being in general independent of the *amplitudes* of vibration, in order that the resulting motion should be as easy and as little violent as possible, it is necessary that for any *given* amounts of the disturbing causes the amplitudes of vibration *should be as small as possible.* Now, the preceding circumstance being fulfilled (but not otherwise), the vertical disturbing force R, and the two vertical moments H and K in any displaced position of the body $\zeta\theta'\theta''$ due to their action or to any other cause, will be opposed respectively (see p. 182 and the notation there employed) by an opposite vertical force $Mg.\left(\frac{A}{V}\right).\zeta$, and by two opposite vertical moments

$$Mg.\left(\frac{I'-Va}{V}\right).\theta' \text{ and } Mg.\left(\frac{I''-Va}{V}\right).\theta'',$$

and it is manifest that the particular values of $\zeta\theta'\theta''$, for which these latter *are in equilibrium with the mean or average values of the former,* will be the *mean* amplitudes of vibration arising from the action of those disturbing causes. In order, therefore, that those mean amplitudes should be as small as possible for given average values of R, H, and K, *their*

coefficients $\left(\frac{A}{V}\right)$, $\left(\frac{I' - Va}{V}\right)$, and $\left(\frac{I'' - Va}{V}\right)$, *should be as large as possible;* that is, V being taken as fixed, A, I', and I'' should be as large, and a as small as possible: the last is obviously that condition of paramount importance necessary alike to the convenience of the motion and to the actual stability of the equilibrium, viz. *that the centre of gravity should be as low down as possible:* the second and third conditions are both to a certain extent contained in the first, viz. *that the area of the plane of floatation should be as great as possible.* If the average value of either of the two vertical moments H or K be very large as compared with the other, the plane of floatation should of course *be as much elongated as possible in the direction of its action,* in order that the corresponding moment of inertia I' or I'' should be as considerable as possible.

And thirdly, which is a case of not unfrequent occurrence, if, in the case of a body whose centre of gravity is not only in the same vertical with that of the area of the plane of floatation but also either actually or nearly coincides with that point, the horizontal be so considerable compared with the vertical disturbing forces as to produce anything like a *finite* displacement of horizontal rotation, then, in addition to the preceding, it is further necessary *that the surface of the body where it enters the fluid all round in the neighbourhood of the plane of floatation should be as nearly vertical as possible;* for, the principal disturbing causes in this case being the two horizontal moments H and K, and their separate or combined tendency being to produce a horizontal rotation round an axis passing through the centre of gravity without any accompanying vertical motion of that point itself, the aforesaid condition is manifestly necessary in order that such should take place to any finite amount without at the same time altering the volume of the displaced fluid (or at least without altering it more than by an insensibly small quantity), and therefore without causing an accompanying abrupt and considerable vertical motion. If the centre of gravity of the body were at any finite distance above or below the plane of floatation, the requisite condition in order to secure similar results*

* To secure the same advantages if the centre of gravity of the body were *below* the plane of floatation, the surface of the body in place of being vertical should actually expand to a certain distance below the surface of the fluid—a circumstance which might be attended in other respects with manifest inconvenience: from this it would seem to follow that the plane of floatation presents sometimes a practical limit to the depression of the centre of gravity of the whole body, below which certain advantages would be lost which in many conceivable cases might possibly be of greater consequence than an increased amount of mere stability of equilibrium.

would still relate to the surface of the body in the neighbourhood of the plane of floatation, and might be easily investigated, but it would not be the same as the above, nor at all of so simple a nature. In either case, however, the additional condition is necessary only for finite but not for infinitely small displacements.

An easy and familiar example, illustrative and at the same time confirmatory of the preceding principles, is afforded by the case of an *ellipsoid* of uniform or variable density, perfect or imperfect in surface, or more generally of *a body of any internal structure bounded by a segment of any convex surface of the second order;* for in this case, the bounding surface X being of the second order, there is no difficulty in seeing that the two auxiliary surfaces Y and Z will always be two other surfaces of the second order, both similar, concentric, and coaxal with X: and from the well-known and familiar properties of surfaces of the second order all the foregoing principles are in their case immediately manifest.

To determine therefore the positions of equilibrium of a floating body terminated by a segment of any convex surface of the second order, *is reduced to draw a normal from a given point, the centre of gravity of the body, to another determinable surface of the second order, concentric, similar, and coaxal with the given surface.* Hence such a body in general can never have more than *eight* positions of equilibrium, and not even so many* except in the case of the *ellipsoid* alone, and that too when the centre of gravity is moreover so conveniently situated as to make the normals from it to the auxiliary ellipsoid be *all* real, for in the case of the *elliptic paraboloid* one of the *real* normals always strikes the auxiliary paraboloid *at an infinite distance,* and corresponds therefore to a direction in which the body being necessarily finite is manifestly imperfect; and in the case of the *hyperboloid of two sheets,* including that of the *cone,* more than one of them in general always strike *the opposite sheet* of the auxiliary hyperboloid, and correspond therefore to the opposite sheet of the given surface, a sheet which manifestly can form no part of the body. In *all* cases, of course, a position is lost for every real normal which would pass through an imperfect part of the bounding surface of the body.

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* In this and what follows, it is to be carefully remembered that the above results refer only to those positions of equilibrium *in which the curvilinear portion of the surface of the body immersed in the fluid is perfect and unbroken;* otherwise the conclusion as stated above would not be accurate.— See Note, page 170.

The three *cylinders of the second order* come under the preceding head, but for them the problem becomes still simpler, for in their case the two auxiliary surfaces Y and Z will manifestly be two other cylinders* of the second order whose transverse sections will be both similar, concentric, and coaxal with that of X, and therefore the normals from any given point to the surface Z will lie all in the plane of the transverse section passing through that point, and will be all normals to the curve of section. To determine therefore the positions of equilibrium of a floating cylinder (of course of finite length) standing on a segment of any curve of the second order, *is reduced to draw a normal from a given point, the centre of gravity of the body, to a determinable conic, similar, concentric, and coaxal with the transverse section of the cylinder passing through that point.* Hence in general such a body can never have more than *four* positions of equilibrium, and not even so many except in the particular case of the *ellipse* alone, and that too only when the centre of gravity is so conveniently situated as to make the normals from it to the auxiliary ellipse be all real; for, in the case of the *parabola,* one of the real normals always meets the auxiliary parabola *at an infinite distance,* and corresponds therefore to a direction in which the cylindrical surface of the body, being necessarily finite, is therefore imperfect; and in the case of the *hyperbola,* including that of *two intersecting right lines,* one of them always strikes *the opposite branch* of the auxiliary hyperbola, and therefore corresponds to the opposite sheet of the given cylinder, a sheet which manifestly can form no part of the body. In the *elliptic* cylinder therefore, supposing it perfect, there will be always either *two* or *four* positions of equilibrium, depending on the position of the centre of gravity according as it is *without or within the evolute of the auxiliary cylinder Z,* while in the *parabolic* and *hyperbolic* cylinders there can *never be more than three* and there *may be but one,* the number as before being determined by the position of the centre of gravity, according as it is within or without the evolute of the auxiliary cylinder Z.

An extremely particular case of the last is that discussed by Poisson (*Traité de Mécanique,* Art. 670), that viz. of *the triangular prism,* and the results just mentioned are the same which have been obtained by him; for the base of such a

* In the present instance Z, properly speaking, is a *curve,* not a *surface,* since the cylinder can not in actual practice be of infinite length. The same obviously will be the case for every body of which the part immersed in the fluid is terminated by a finite portion of any developable surface.

prism being necessarily terminated at each angle is a segment of a single branch of three different hyperbolas, of which the three auxiliary conics corresponding to the three different angles consist each of a portion of one branch of an hyperbola having for its asymptotes the sides of the corresponding angle; hence, from the above, we see that *there will be either one or three positions of equilibrium corresponding to each angle*, according as the centre of gravity is within or without the evolute of the auxiliary hyperbola belonging to that angle, *but never four for any angle*, the fourth for each corresponding always to the opposite branch of the hyperbola in the region corresponding to which the body, considered as a cylinder, is necessarily imperfect—results exactly the same as those obtained by Poisson. (*Traité de Mécanique*, Art. 607).

Trinity College, Dublin, Feb. 19, 1849.

ON THE CONE CIRCUMSCRIBING A SURFACE OF THE m^{th} ORDER.

By the Rev. GEORGE SALMON.

1. "If such a cone be cut by any plane, the points of inflexion of the section will be the projections of the points where the cone meets the parabolic curve on the surface." For, a point of inflexion is a point at which two consecutive tangents coincide; but the tangent planes to the cone at these points are also tangent planes to the surface, and we have seen (*Math. Journal*, vol. III. p. 45) that the parabolic curve is the locus of all the points for which two consecutive tangent planes to the surface coincide.

2. "The number of such points of inflexion will be

$$= 4m(m-1)(m-2)."$$

For the points where the cone meets the surface lie on a surface of the $(m-1)^{\text{st}}$ degree, and the parabolic curve lies on a surface of the $4(m-2)$ degree. (*Journal*, vol. II. p. 74.)

The same result may be otherwise obtained from the ordinary formula for the number of points of inflexion on any curve

$$i = 3n(n-2) - 8x - 6y,$$

where x is the number of the cusps, and y of the double

points on the curve: but in the circumscribing cone (*Journal*, vol. II. p. 66) $n = m^2 - m$,

$$x = m(m-1)(m-2), \quad y = \frac{m(m-1)(m-2)(m-3)}{2}.$$

Substituting these values in the equation just given, we get the same value as before for the number of points of inflexion.

3. "If we form the reciprocal of the given surface of the m^{th} degree, this surface will have a cuspidal line of the degree $4m(m-1)(m-2)$, this curve being the reciprocal of the developable which touches the given surface along the parabolic curve."

A cuspidal line is one at every point of which can be drawn two coincident tangent planes; to this system of points must therefore correspond a system of planes which touch the given surface in two coincident points: but this is the characteristic of the tangent planes at the points of the parabolic curve.

4. "If the given surface have a double line of the degree a, the number of cuspidal edges on the circumscribing cone will be diminished by $3a(m-2)$."

The circumscribing cone in this case breaks up into a double cone passing through the curve a, and a cone of the degree $m^2 - m - 2a$, which last is the circumscribing cone proper. The number of cuspidal edges in the total complex cone is the same as before; the number therefore of cuspidal edges on the proper circumscribing cone will be less than in the general case by the number which lie on the cone a. It was proved (*Journal*, vol. II. p. 69) that a cuspidal edge on the circumscribing cone takes place for every point where the curve of contact meets the second polar surface, whose order is $m - 2$. Each of the $a(m-2)$ points where the curve a meets this surface counts for three, because at these points not only a is a double line on the surface, but also the first polar surface has its tangent plane coincident with one of the tangent planes to the surface. This will be readily understood by considering the case where $m = 3$, $a = 1$; the second polar surface then meets the surface in a plane curve having a double point, and meets the first polar in a conic passing through the same point, and having for its tangent one of the tangents at the double point. The conic therefore and the curve of the 3rd degree only meet in three other points.

5. "If the given surface have a double line of the degree a such that any cone standing on the curve a has h double edges; then the number of ordinary double lines on the circumscribing cone is diminished by

$$2a(m-2)(m-3)-4h."$$

Let the number of double edges on the circumscribing cone proper be x; and the number of edges of the cone a, which elsewhere touch the surface, be z; then it is not difficult to see that the double edges of the total complex cone

$$=4h+2z+x,$$

and that therefore x is less than in the general case by

$$4h+2z.$$

I determine z as follows. In the case where $a=1$, I find that the circumscribing cone touches the cone a in the lines to the $a(m-2)$ points determined in the last paragraph; that it meets it also in the lines to the *cuspidal points* of a (*Journal*, vol. II. p. 72); and that the remaining intersections are the lines z. I assume this to be generally true.

It was proved also in the case where a is the complete intersection of two surfaces, that the number of cuspidal points is

$$a(2m-2a-2)+4h.$$

I assume this likewise to be generally true.

We have then the equation

$$a(m^2-m-2a)=z+2a(m-2)+a(2m-2a-2)+4h;$$

hence $$z=a(m^2-5m+6)-4h;$$

and therefore $$2z+4h=2a(m-2)(m-3)-4h.$$

The truth of this result of course depends on the correctness of the assumptions above made.

6. We can thus determine the diminution produced in the degree of the reciprocal surface by such a double line on the original surface.

The degree of the circumscribing cone being reduced from m^2-m to m^2-m-2a, the degree of its reciprocal will be reduced from

$$(m^2-m)(m^2-m-1) \text{ to } (m^2-m-2a)(m^2-m-2a-1),$$

or by $$a(4m^2-4m-4a-2).$$

We must deduct from this the diminution consequent on the loss of cusps in the circumscribing cone $=9a(m-2)$, and

also that consequent on the loss of double edges

$$= 4a\{(m-2)(m-3)\} - 8h:$$

the diminution then on the degree of the reciprocal surface is

$$a(7m - 4a - 8) + 8h.$$

This coincides with the formula before given in this *Journal*, but can be applied to some cases to which the latter is not applicable: for example, it would follow from the present formula that the reciprocal of a surface of the fourth degree having two double right lines not in the same plane, is another similar surface of the fourth degree; that the reciprocal of a surface of the fourth degree having as an ordinary double line a proper curve of the third degree, is also a similar surface of the fourth degree, &c.

7. I do not find that ordinary double points on the curve a have any effect on the circumscribing cone. It will be otherwise, however, if there be on that curve a triple point which is also a triple point on the surface. Three then of the points determined in § 4 will coincide; but instead of the point in which they meet producing a diminution of 9 in the number of cuspidal edges of the circumscribing cone, I find that it will only produce a diminution of 6. This appeared from an examination of the particular surface

$$\sqrt{x} + \sqrt{y} + \sqrt{z} + \sqrt{w} = 0,$$

which has three double lines meeting in a triple point. The result therefore of § 4 will in this case be modified to

$$3a(m-2) - 3t;$$

and if we suppose the triple point to have no effect on the diminution of the double points of the tangent cone, the degree of the reciprocal surface will be diminished by

$$a(7m - 4a - 8) + 8h + 9t.$$

Trinity College, Dublin, March 26, 1849.

ON GEODESIC LINES TRACED ON A SURFACE OF THE SECOND DEGREE.

By Andrew S. Hart, F.T.C.D.

In the last No. of this *Journal* I made use of Mr. Roberts' equation $P \sin \omega = y$ to investigate the form of any geodesic line which passes through an umbilic of an ellipsoid. The object of the present article is to extend the same method of investigation to all geodesic lines traced on a surface of the second degree.

Let A be any such surface, and conceive a group of geodesic lines on this surface touching its intersection with any given confocal surface B. Let OT, $O'T'$ be two of these lines, O and O' being two consecutive points on the intersection of A and B. Through T draw a tangent to the line OT, it will touch the surface B; let P be the point of contact, and if PT and TT' be parallel to two conjugate diameters of the surface A, the tangent at T' to $O'T'$ will lie in the same plane with PT and will meet it at a point indefinitely close to P.

Now since the lines OT and $O'T'$ intersect at an indefinitely small angle $d\omega$, the perpendicular distance of T' from OT may be denoted by $Pd\omega$, and we might find the value of P by the same method which Mr. Roberts applied in a former No. of this *Journal* to the lines which pass through an umbilic, or more briefly thus.

Let the angle $TPT' = d\phi$, and let $TP = t$, then $td\phi = Pd\omega$; but if T_1, T_1' be two points indefinitely close to T and T' on the lines OT, $O'T'$, and if the tangents at these points meet at P_1 infinitely close to their points of contact with B, and if

$$P_1T_1 = t', \quad T_1P_1T_1' = d\phi', \quad TT_1 = d\rho,$$

we have
$$(t' + d\rho)\, d\phi' = Pd\omega = td\phi.$$

But if D and D' be the diameters of B parallel to t and t', we have $t' + d\rho : t :: D' : D$, therefore $Dd\phi = D'd\phi'$: and in like manner, if D'' and $d\phi''$ be the values of D and $d\phi$ corresponding to the next point on the line OT, $D''d\phi'' = D'd\phi'$, therefore this product is invariable; and therefore if D_0 be the diameter of B parallel to the tangent at O, $Dd\phi = D_0 d\omega$, but $td\phi = Pd\omega$, therefore $P = \dfrac{D_0}{D}\, t$, or P is a mean proportional between the segments of a chord of B drawn through T parallel to the tangent at O. If O were an umbilic, B would coincide with the plane of the umbilics, and this value of P would agree with that already given by Mr. Roberts.

From the equation $Dd\phi$ = constant, it follows that if the two lines OT, $O'T'$ be produced to touch the intersection of A and B a second time, and if D_1 be the diameter of B parallel to the tangent at the second point of contact, and $d\omega_1$ the corresponding value of $d\omega$, $D_0 d\omega = D_1 d\omega_1$, whence it follows that $\int Dd\omega$ from $D = D_0$ to $D = D_1$ is constant for all geodesic lines of this group. Also if two lines of the group intersect on a given line of curvature, and two consecutive lines intersect at an adjacent point on the same line of curvature, $Pd\omega$ and therefore also $Dd\omega$ will be the same for each pair of consecutive lines, therefore $\int Dd\omega$ between two points of contact with the line of intersection of A and B is constant if the geodesic tangents at these points intersect on a given line of curvature.

It is easy to determine the value of this constant by considering the lines which intersect on a principal section of the surface A, but the result does not appear to possess much interest. It is also evident from the theorem of Joachimsthal that we may substitute for D in each of these equations the reciprocal of the perpendicular from the centre of the surface on the osculating plane.

It is also evident that if the surface A was intersected by any other surface similar to B, the value of P would be the same at all intersections of one of these geodesic lines with this surface; that at each intersection with the asymptotic cone $P = D$; and that if the osculating plane passes through the centre of the given surface $d\phi = 0$, and therefore P is a maximum.

It follows immediately from the value of P, that if any number of geodesic tangents to a given line of curvature form a polygon, and if all the angles of this polygon except one move upon lines of curvature, this angle will also move upon a line of curvature, and the perimeter of the polygon will be constant if all the lines of curvature be of the same kind. For if O, O', O'', &c. be the points of contact, and A, B, C, &c. the initial positions of the angles of the polygon, and a, b, c, &c. the next positions of these points when the polygon begins to move, then let the angles $AOa = d\omega$, $BO'b = d\omega'$, &c., and let the values of P corresponding to the segments AO and OB be P and Q, the values corresponding to BO' and $O'C$, P' and Q', &c., and let the diameters of the confocal surface which passes through the line of curvature be D parallel to the tangent at O, D' parallel to the tangent at O', &c.; then we have

$$Q : P' :: D : D', \quad Q' : P'' :: D' : D'', \text{ \&c.};$$

and since Aa, Bb, &c. bisect the angles of the polygon

$$Qd\omega = P'd\omega', \quad Q'd\omega' = P''d\omega'', \text{ \&c.},$$

therefore $Dd\omega = D'd\omega' = D''d\omega''$, and so on to the last angle of the polygon; therefore it also moves on the bisector of the angle. Q. E. D.

It is evident that if all the lines of curvature be of the same kind, since each pair of tangents exceeds the intercepted arch by a constant quantity, the entire perimeter is constant. A particular case of this proposition has been already published by Mr. Roberts in *Liouville's Journal*, where the polygon is a quadrilateral, two of whose angles move on the same line of curvature, and the other lines of curvature are of the same kind.

It is hardly necessary to observe that the foregoing propositions are also true of confocal plane conics, D, D', &c. being the diameters of the interior conic, and the geodesic lines being rectilinear tangents to this conic.

Trinity College, Dublin, March 14, 1849.

ON ATTRACTIONS, AND ON CLAIRAUT'S THEOREM.

By G. G. Stokes.

Clairaut's Theorem is usually deduced as a consequence of the hypothesis of the original fluidity of the earth, and the near agreement between the numerical values of the earth's ellipticity, deduced independently from measures of arcs of the meridian and from pendulum experiments, is generally considered as a strong confirmation of the hypothesis. Although this theorem is usually studied in connexion with the hypothesis just mentioned, it ought to be observed that Laplace, without making any assumption respecting the constitution of the earth, except that it consists of nearly spherical strata of equal density, and that its surface may be regarded as covered by a fluid, has established a connexion between the form of the surface and the variation of gravity, which in the particular case of an oblate spheroid gives directly Clairaut's Theorem.* If, however, we merely assume,

* See the *Mécanique Céleste*, Liv. III., or the reference to it in Pratt's *Mechanics*, Chap. *Figure of the Earth.*

as a matter of observation, that the earth's surface is a surface of equilibrium, (the trifling irregularities of the surface being neglected), that is to say that it is perpendicular to the direction of gravity, then, independently of any particular hypothesis respecting the state of the interior, or any theory but that of universal gravitation, there exists a necessary connexion between the form of the surface and the variation of gravity along it, so that the one being given the other follows. In the particular case in which the surface is an oblate spheroid of small eccentricity, which the measures of arcs shew to be at least very approximately the form of the earth's surface, the variation of gravity is expressed by the equation which is arrived at on the hypothesis of original fluidity. I am at present engaged in preparing a paper on this subject for the Cambridge Philosophical Society: the object of the following pages is to give a demonstration of Clairaut's Theorem, different from the one there employed, which will not require a knowledge of the properties of the functions usually known by the name of Laplace's Coefficients. It will be convenient to commence with the demonstration of a few known theorems relating to attractions, the law of attraction being that of the inverse square of the distance.*

Preliminary Propositions respecting Attractions.

Prop. i. To express the components of the attraction of any mass in three rectangular directions by means of a single function.

Let m' be the mass of an attracting particle situated at the point P', the unit of mass being taken as is usual in central forces, m the mass of the attracted particle situated at the point P, x', y', z' the rectangular coordinates of P' referred to any origin, x, y, z those of P; X, Y, Z the components of

* My object in giving these demonstrations is simply to enable a reader who may not have attended particularly to the theory of attractions to follow with facility the demonstration here given of Clairaut's Theorem. In speaking of the theorems as "known" I have, I hope, sufficiently disclaimed any pretence at originality. In fact, not one of the "propositions respecting attractions" is new, although now and then the demonstrations may differ from what have hitherto been given. With one or two exceptions, these propositions will all be found in a paper by Gauss, of which a translation is published in the third volume of *Taylor's Scientific Memoirs*, p. 153. The demonstrations here given of Propositions vi. and vii. are the same as Gauss's; that of Prop. v., though less elegant than Gauss's, appears to me more natural. The ideas on which it depends render it closely allied to a paper by Professor Thomson, in the third volume of this *Journal* (Old Series) p. 71. Prop. ix. is given merely for the sake of exemplifying the application of the same mode of proof to a theorem of Gauss's.

the attraction of m' on m, measured as accelerating forces, and considered positive when they tend to increase x, y, z; then, if $PP' = r'$,

$$X = \frac{m'}{r'^3}(x' - x), \quad Y = \frac{m'}{r'^3}(y' - y), \quad Z = \frac{m'}{r'^3}(z' - z).$$

Since $$r'^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2,$$

we have $r'\dfrac{dr'}{dx} = -(x' - x)$; whence $X = -\dfrac{m'}{r'}\dfrac{dr'}{dx} = \dfrac{d}{dx}\dfrac{m'}{r'}$;

with similar equations for Y and Z.

If instead of a single particle m' we have any number of attracting particles $m', m'' \ldots$ situated at the points (x', y', z'), $(x'', y'', z'') \ldots$, and if we put

$$\frac{m'}{r'} + \frac{m''}{r''} + \ldots = \Sigma\frac{m'}{r'} = V \ldots\ldots\ldots (1),$$

we get

$$X = \frac{d}{dx}\left(\frac{m'}{r'} + \frac{m''}{r''} + \ldots\right) = \frac{dV}{dx}; \text{ similarly } Y = \frac{dV}{dy}, Z = \frac{dV}{dz} \ldots (2).$$

If instead of a set of distinct particles we have a continuous attracting mass M', and if we denote by dm' a differential element of M', and replace (1) by

$$V = \iiint\frac{dm'}{r'} \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

equations (2) will still remain true, provided at least P be external to M'; for it is only in that case that we are at liberty to consider the continuous mass as the limit of a set of particles which are all situated at finite distances from P. It must be observed that should M' occupy a closed shell, within the inner surface of which P is situated, P must be considered as external to the mass M'. Nevertheless, even when P lies within M', or at its surface, the expressions for V and $\dfrac{dV}{dx}$, namely $\iiint\dfrac{dm'}{r'}$ and $\iiint(x' - x)\dfrac{dm'}{r'^3}$, admit of real integration, defined as a limiting summation, as may be seen at once on referring M' to polar coordinates originating at P; so that the equations (2) still remain true.

Prop. II. To express the attraction resolved along any line by means of the function V.

Let s be the length of the given line measured from a fixed point up to the point P; λ, μ, ν, the direction-cosines of the tangent to this line at P, F the attraction resolved

along this tangent; then

$$F = \lambda X + \mu Y + \nu Z = \lambda \frac{dV}{dx} + \mu \frac{dV}{dy} + \nu \frac{dV}{dz}.$$

Now if we restrict ourselves to points lying in the line s, V will be a function of s alone; or we may regard it as a function of x, y, and z, each of which is a function of s; and we shall have, by Differential Calculus,

$$\frac{dV}{ds} = \frac{dV}{dx}\frac{dx}{ds} + \frac{dV}{dy}\frac{dy}{ds} + \frac{dV}{dz}\frac{dz}{ds};$$

and since $\frac{dx}{ds} = \lambda$, $\frac{dy}{ds} = \mu$, $\frac{dz}{ds} = \nu$, we get

$$F = \frac{dV}{ds} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4).$$

Prop. III. To examine the meaning of the function V.

This function is of so much importance that it will be well to dwell a little on its meaning.

In the first place it may be observed that the equation (1) or (3) contains a physical definition of V, which has nothing to do with the system of coordinates, rectangular, polar, or any other, which may be used to define algebraically the positions of P and of the attracting particles. Thus V is to be contemplated as a function of the position of P in space, if such an expression may be allowed, rather than as a function of the coordinates of P; although, in consequence of its depending upon the position of P, V will be a function of the coordinates of P, of whatever kind they may be.

Secondly, it is to be remarked that although an attracted particle has hitherto been conceived as situated at P, yet V has a definite meaning, depending upon the position of the point P, whether any attracted matter exist there or not. Thus V is to be contemplated as having a definite value at each point of space, irrespective of the attracted matter which may exist in some places.

The function V admits of another physical definition which ought to be noticed. Conceive a particle whose mass is m to move along any curve from the point P_0 to P. If F be the attraction of M' resolved along a tangent to m's path, reckoned as an accelerating force, the moving force of the attraction resolved in the same direction will be mF, and therefore the work done by the attraction while m describes the elementary arc ds will be ultimately $mFds$, or

by (4) $m \dfrac{dV}{ds} ds$. Hence the whole work done as m moves from P_0 to P is equal to $m(V - V_0)$, V_0 being the value of V at P_0. If P_0 be situated at an infinite distance, V_0 vanishes, and the expression for the work done becomes simply mV. Hence V might be called the *work of the attraction, referred to a unit of mass of the attracted particle;* but besides that such a name would be inconveniently long, a recognized name already exists. The function V is called the *potential* of the attracting mass.*

The first physical definition of V is peculiar to attraction according to the inverse square of the distance. According to the second, V is regarded as a particular case of the more general function whose partial differential coefficients with respect to x, y, z are equal to the components of the accelerating force; a function which exists whenever $Xdx + Ydy + Zdz$ is an exact differential.

PROP. IV. If S be any closed surface to which all the attracting mass is external, dS an element of S, dn an element of the normal drawn outwards at dS, then

$$\iint \frac{dV}{dn} dS = 0 \quad \ldots\ldots\ldots\ldots (5),$$

the integral being taken throughout the whole surface S.

Let m' be the mass of any attracting particle which is situated at the point P', P' being by hypothesis external to S. Through P' draw any right line L cutting S, and produce it indefinitely in one direction from P'. The line L will in general cut S in two points; but if the surface S be re-entrant, it may be cut in four, six, or any even number of points. Denote the points of section, taken in order, by P_1, P_2, P_3, &c., P_1 being that which lies nearest to P'. With P' for vertex, describe about the line L a conical surface containing an infinitely small solid angle α, and denote by $A_1, A_2 \ldots$ the areas which it cuts out from S about the points $P_1, P_2 \ldots$. Let $\theta_1, \theta_2 \ldots$ be the angles which the normals drawn outwards at $P_1, P_2 \ldots$ make with the line L, taken in the direction from P_1 to P'; $N_1, N_2 \ldots$ the attractions of m' at $P_1, P_2 \ldots$ resolved along the normals; $r_1, r_2 \ldots$ the distances of $P_1, P_2 \ldots$ from P'. It is

* [The term "potential," as used in the theory of Electricity, may be defined in the following manner: "The potential at any point P, in the neighbourhood of electrified matter, is the amount of work that would be necessary to remove a small body charged with a unit of negative electricity from that position to an infinite distance."—W. T.]

evident that the angles $\theta_1, \theta_2 \dots$ will be alternately acute and obtuse. Then we have

$$N_1 = \frac{m'}{r_1^2} \cos \theta_1, \quad N_2 = -\frac{m'}{r_2^2} \cos (\pi - \theta_2) \text{ \&c.}$$

We have also in the limit

$$A_1 = ar_1^2 \sec \theta_1, \quad A_2 = ar_2^2 \sec (\pi - \theta_2), \text{ \&c.};$$

and therefore $N_1A_1 = am'$, $N_2A_2 = -am'$, $N_3A_3 = am'$, &c.;

and therefore, since the number of points $P_1, P_2 \dots$ is even,

$$N_1A_1 + N_2A_2 + N_3A_3 + N_4A_4 \dots = am' - am' + am' - am' \dots = 0.$$

Now the whole solid angle contained within a conical surface described with P' for vertex so as to circumscribe S may be divided into an infinite number of elementary solid angles, to each of which the preceding reasoning will apply; and it is evident that the whole surface S will thus be exhausted. We have therefore

$$\text{limit of } \Sigma NA = 0;$$

or, by the definition of an integral,

$$\iint NdS = 0.$$

The same will be true of each attracting particle m'; and therefore if N refer to the attraction of the whole attracting mass, we shall still have $\iint NdS = 0$. But by (4) $N = \dfrac{dV}{dn}$, which proves the proposition.

Prop. v. If V be equal to zero at all points of a closed surface S, which does not contain any portion of the attracting mass, it must be equal to zero at all points of the space T contained within S.

For if not, V must be either positive or negative in at least a certain portion of the space T, and therefore must admit of at least one positive or negative maximum value V_1. Call the point, or the assemblage of connected points, at which V has its maximum value V_1, T_1. It is to be observed, first, that T_1 may denote either a space, a surface, a line, or a single point; secondly, that should V happen to have the same value V_1 at other points within T', such points must not be included in T_1. Then, all round T_1, V is decreasing, positively or negatively according as V_1 is positive or negative. Circumscribe a closed surface S_1 around T_1, lying wholly within S, which is evidently possible. Then if S_1 be drawn sufficiently close round T_1, V will be increasing

in passing outwards across S_1;* and therefore, if n_1 denote a normal drawn outwards at the element dS_1 of S_1, $\frac{dV}{dn_1}$ will be negative or positive according as V_1 is positive or negative, and therefore $\iint \frac{dV}{dn_1}\, dS_1$, taken throughout the whole surface S_1, will be negative or positive, which is contrary to Prop. IV. Hence V must be equal to zero throughout the space T.

COR. 1. If V be equal to a constant A at all points of the surface S, it must be equal to A at all points within S. For it may be proved just as before that V cannot be either greater or less than A within S.

COR. 2. If V be not constant throughout the surface S, and if A be its greatest, and B its least value in that surface, V cannot anywhere within S be greater than A nor less than B.

COR. 3. All these theorems will be equally true if the space T extend to infinity, provided that instead of the value of V at the bounding surface of T we speak of the value of V at the surface by which T is partially bounded, and its limiting value at an infinite distance in T. This limiting value might be conceived to vary from one direction to another. Thus T might be the infinite space lying within one sheet of a cone, or hyperboloid of one sheet, or the infinite space which lies outside a given closed surface S, which contains within it all the attracting mass. On the latter supposition, if V be equal to zero throughout S, and vanish at an infinite distance, V must be equal to zero everywhere outside S. If V vanish at an infinite distance, and range between the limits A and B at the surface S, V cannot anywhere outside S lie beyond the limits determined by the two extremes of the three quantities A, B, and 0.

PROP. VI. At any point (x, y, z) external to the attracting mass, the potential V satisfies the partial differential equation

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = 0 \ldots\ldots\ldots\ldots (6).$$

* It might, of course, be possible to prevent this by drawing S_1 sufficiently puckered, but S_1 is supposed not to be so drawn. Since V is decreasing from T_1 outwards, if we consider the loci of the points where V has the values V_2, V_3, V_4 . . . decreasing by infinitely small steps from V_1, it is evident that in the immediate neighbourhood of T_1 these loci will be closed surfaces, each lying outside the preceding, the first of which ultimately coincides with T_1 if T_1 be a point, a line, or a surface, or with the surface of T_1 if T_1 be a space. If now we take for S_1 one of these "surfaces of equilibrium," or any surface cutting them at acute angles, what was asserted in the text respecting S_1 will be true.

For if V' denote the potential of a single particle m', we have, employing the notation of Prop. I.,

$$V' = \frac{m'}{r'}, \quad \frac{dV'}{dx} = -\frac{m'}{r'^2}\frac{dr'}{dx} = \frac{m'}{r'^3}(x' - x), \quad \frac{d^2V'}{dx^2} = \frac{3m'}{r'^5}(x' - x)^2 - \frac{m'}{r'^3},$$

with similar expressions for $\frac{d^2V'}{dy^2}$ and $\frac{d^2V'}{dz^2}$; and therefore V' satisfies (6). This equation will be also satisfied by the potentials V'', V'''.. of particles m'', m'''.. situated at finite distances from the point (x, y, z), and therefore by the potential V of all the particles, since $V = V' + V'' + V''' + \ldots$ Now, by supposing the number of particles indefinitely increased, and their masses, as well as the distances between adjacent particles, indefinitely diminished, we pass in the limit to a continuous mass, of which all the points are situated at finite distances from the point (x, y, z). Hence the potential V of a continuous mass satisfies equation (6) at all points of space to which the mass does not reach.

SCHOLIUM to Prop. V. Although the equations (5) and (6) have been proved independently of each other from the definition of a potential, either of these equations is a simple analytical consequence of the other.* Now the only property

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* The equation (6) will be proved by means of (5) further on, (Prop. VIII.), or rather an equation of which (6) is a particular case, by means of an equation of which (5) is a particular case. Equation (5) may be proved from (6) by a known transformation of the equation $\iiint \nabla V\, dx\, dy\, dz = 0$, where ∇V denotes the first member of (6), and the integration is supposed to extend over the space T. For, taking the first term in ∇V, we get

$$\iiint \frac{d^2V}{dx^2}\, dx\, dy\, dz = \iint \left(\frac{dV}{dx}\right)_{\prime\prime} dy\, dz - \iint \left(\frac{dV}{dx}\right)_{\prime} dy\, dz,$$

where $\left(\frac{dV}{dx}\right)_{\prime\prime}$, $\left(\frac{dV}{dx}\right)_{\prime}$ denote the values of $\frac{dV}{dx}$ at the points where S is cut by a line drawn parallel to the axis of x through the point whose coordinates are $0, y, z$. Now if λ be the angle between the normal drawn outwards at the element of surface dS and the axis of x,

$$\iint \left(\frac{dV}{dx}\right)_{\prime\prime} dy\, dz = \iint \frac{dV}{dx} \cos\lambda\, dS, \quad \iint \left(\frac{dV}{dx}\right)_{\prime} dy\, dz = \iint \frac{dV}{dx} \cos(\pi - \lambda)\, dS,$$

where the first integration is to be extended over the portion of S which lies to the positive side of the curve of contact of S and an enveloping cylinder with its generating lines parallel to the axis of x, and the second integration over the remainder of S. If then we extend the integration over the whole of the surface S, we get

$$\iiint \frac{d^2V}{dx^2}\, dx\, dy\, dz = \iint \frac{dV}{dx} \cos\lambda . dS.$$

of a potential assumed in Prop. v. is, that it is a quantity which varies continuously within the space T, and satisfies the equation (5) for any closed surface drawn within T. Hence Prop. v., which was enunciated with respect to the potential of a mass lying outside T, is equally true with respect to any continuously varying quantity which within the space T satisfies the equation (6). It should be observed that a quantity like r^{-1} is not to be regarded as such, if r denote the distance of the point (x, y, z) from a point $P_{,}$ which lies within T, because r^{-1} becomes infinite at $P_{,}$.

Clairaut's Theorem.

1. Although the earth is really revolving about its axis, so that all problems relating to the relative equilibrium of the earth itself and the bodies on its surface are really dynamical problems, we know that they may be treated statically by introducing, in addition to the attraction, that fictitious force which we call the centrifugal force. The force of gravity is the resultant of the attraction and the centrifugal force; and we know that this force is perpendicular to the general surface of the earth. In fact, by far the larger portion of the earth's surface is covered by water, the equilibrium of which requires, according to the principles of hydrostatics, that its surface be perpendicular to the direction of gravity; and the elevation of the land above the level of the sea, or at least the elevation of large tracts of land, is but trifling compared with the dimensions of the earth. We may therefore regard the earth's surface as a surface of equilibrium.

2. Let the earth be referred to rectangular axes, the axis of z coinciding with the axis of rotation. Let V be the potential of the mass, ω the angular velocity, X, Y, Z the components of the whole force at the point (x, y, z); then

$$X = \frac{dV}{dx} + \omega^2 x, \quad Y = \frac{dV}{dy} + \omega^2 y, \quad Z = \frac{dV}{dz}.$$

Making a similar transformation with respect to the two remaining terms of ∇V, and observing that if μ, ν be for y, z what λ is for x,

$$\cos\lambda \frac{dV}{dx} + \cos\mu \frac{dV}{dy} + \cos\nu \frac{dV}{dz} = \frac{dV}{dn},$$

we obtain equation (5).

If V be any continuously varying quantity which within the space T satisfies the equation $\nabla V = 0$, it may be proved that it is always possible to distribute attracting matter outside T in such a manner as to produce within T a potential equal to V.

Now the general equation to surfaces of equilibrium is

$$\int(Xdx + Ydy + Zdz) = \text{const.},$$

and therefore we must have at the earth's surface

$$V + \frac{\omega^2}{2}(x^2 + y^2) = c \quad \ldots\ldots\ldots\ldots(7),$$

where c is an unknown constant. Moreover V satisfies the equation (6) at all points external to the earth, and vanishes at an infinite distance. But these conditions are sufficient to determine V at all points of space external to the earth. For if possible let V admit of two different values V_1, V_2 outside the earth, and let $V_1 - V_2 = V'$. Since V_1 and V_2 have the same value $c - \frac{\omega^2}{2}(x^2 + y^2)$ at the surface, V' vanishes at the surface; and it vanishes likewise at an infinite distance, and therefore by Prop. v. $V' = 0$ at all points outside the earth. Hence if the form of the surface be given, V is determinate at all points of external space, except so far as relates to the single arbitrary constant c which is involved in its complete expression.

3. Now it appears from measures of arcs of the meridian, that the earth's surface is represented, at least very approximately, by an oblate spheroid of small ellipticity, having its axis of figure coinciding with the axis of rotation. It will accordingly be more convenient to refer the earth to polar, than to rectangular coordinates. Let the centre of the surface be taken for origin; let r be the radius vector, θ the angle between this radius and the axis of z, ϕ the angle between the plane passing through these lines and the plane xz. Then if the square of the ellipticity be neglected, the equation to the surface may be put under the form

$$r = a(1 - \varepsilon\cos^2\theta) \quad \ldots\ldots\ldots\ldots(8);$$

and from (7) we must have at the surface

$$V + \frac{\omega^2}{2}a^2\sin^2\theta = c \quad \ldots\ldots\ldots\ldots(9).$$

If we denote for shortness the equation (6) by $\nabla V = 0$, we have, by transformation to polar coordinates,*

$$\nabla V = \frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dV}{d\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{d^2V}{d\phi^2} = 0 \quad \ldots\ldots\ldots\ldots(10).$$

* *Cambridge Mathematical Journal*, vol. I. (*Old Series*) p. 122, or O'Brien's Tract on the Figure of the Earth, p. 12.

4. The form of the equations (8) and (9) suggests the occurrence of terms of the form $\psi(r) + \chi(r)\cos^2\theta$ in the value of V. Assume then

$$V = \psi(r) + \chi(r)\cos^2\theta + w \ldots\ldots\ldots (11).$$

We are evidently at liberty to make this assumption, on account of the indeterminate function w. Now if we observe that

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d.\cos^2\theta}{d\theta}\right) = 2 - 6\cos^2\theta,$$

we get from (10) and (11)

$$\psi''(r) + \frac{2}{r}\psi'(r) + \frac{2}{r^2}\chi(r) + \left\{\chi''(r) + \frac{2}{r}\chi'(r) - \frac{6}{r^2}\chi(r)\right\}\cos^2\theta + \nabla w = 0$$
$$\ldots\ldots\ldots\ldots (12).$$

If now we determine the functions ψ, χ from the equations

$$\psi''(r) + \frac{2}{r}\psi'(r) + \frac{2}{r^2}\chi(r) = 0 \ldots\ldots\ldots (13),$$

$$\chi''(r) + \frac{2}{r}\chi'(r) - \frac{6}{r^2}\chi(r) = 0 \ldots\ldots\ldots (14),$$

we shall have $\nabla w = 0$.

By means of (14), equation (13) may be put under the form

$$\psi''(r) + \frac{2}{r}\psi'(r) = -\tfrac{1}{3}\left\{\chi''(r) + \frac{2}{r}\chi'(r)\right\};$$

and therefore $\psi(r) = -\frac{1}{3}\chi(r)$ is a particular integral of (13). The equations (14), and (13) when deprived of its last term, are easily integrated, and we get

$$\psi(r) = \frac{A}{r} + B - \tfrac{1}{3}\chi(r), \quad \chi(r) = \frac{C}{r^3} + Dr^2 \ldots\ldots (15).$$

Now V vanishes at an infinite distance; and the same will be the case with w provided we take $B = 0$, $D = 0$, when we get from (11) and (15)

$$V = \frac{A}{r} + \frac{C}{r^3}(\cos^2\theta - \tfrac{1}{3}) + w \ldots\ldots\ldots (16).$$

5. It remains to satisfy (9). Now this equation may be satisfied, so far as the large terms are concerned, by means of the constant A, since θ appears only in the small terms. We have a right then to assume C to be a small quantity of

the first order. Substituting in (16) the value of r given by (8), putting the resulting value of V in (9), and retaining the first order only of small quantities, we get

$$\frac{A}{a}(1+\varepsilon\cos^2\theta)+\frac{C}{a^3}(\cos^2\theta-\tfrac{1}{3})+w_{,}+\frac{\omega^2}{2}a^2(1-\cos^2\theta)=c\ldots(17),$$

$w_{,}$ being the value of w at the surface of the earth. Now the constants A and C allow us to satisfy this equation without the aid of $w_{,}$. We get by equating to zero the sum of the constant terms, and the coefficient of $\cos^2\theta$,

$$\left.\begin{aligned}\frac{A}{a}-\frac{C}{3a^3}+\frac{\omega^2a^2}{2}&=c\\ \frac{A\varepsilon}{a}+\frac{C}{a^3}-\frac{\omega^2a^2}{2}&=0\end{aligned}\right\}\ldots\ldots\ldots\ldots(18).$$

These equations combined with (17) give $w_{,}=0$. Now we have seen that w satisfies the equation $\nabla w=0$ at all points exterior to the earth, and that it vanishes at an infinite distance; and since it also vanishes at the surface, it follows from Prop. v. that it is equal to zero every where without the earth.

It is true that $w_{,}$ is not strictly equal to zero, but only to a small quantity of the second order, since quantities of that order are omitted in (17). But it follows from Prop. v. Cor. 3, that if w', w'' be respectively the greatest and least values of $w_{,}$, w cannot anywhere outside the earth lie beyond the limits determined by the two extremes of the three quantities w', w'', and 0, and therefore must be a small quantity of the second order; and since we are only considering the potential at external points, we may omit w altogether.

If E be the mass of the earth, the potential at a very great distance r is ultimately equal to $\dfrac{E}{r}$. Comparing this with the equation obtained from (16) by leaving out w, we get

$$A=E.$$

The first of equations (18) serves only to determine c in terms of E, and c is not wanted. The second gives

$$C=-Ea^2\varepsilon+\tfrac{1}{2}\omega^2a^5,$$

whence, we get from (16)

$$V=\frac{E}{r}-\left(\frac{E\varepsilon}{a}-\tfrac{1}{2}\omega^2a^2\right)\frac{a^3}{r^3}(\cos^2\theta-\tfrac{1}{3})\ldots.(19).$$

6. If g be the force of gravity at any point of the surface, ν the angle between the vertical and the radius vector drawn from the centre, $g \cos \nu$ will be the resolved part of gravity along the radius vector; and we shall have

$$g \cos \nu = -\frac{d}{dr}\left(V + \frac{\omega^2}{2} r^2 \sin^2 \theta\right) \ldots\ldots (20).$$

where after differentiation r is to be put equal to the radius vector of the surface. Now ν is a small quantity of the first order, and therefore $\cos \nu$ may be replaced by 1, whence we get from (8), (19), and (20),

$$g = \frac{E}{a^2}(1 + 2\varepsilon \cos^2 \theta) - 3\left(\frac{E\varepsilon}{a^2} - \tfrac{1}{2}\omega^2 a\right)(\cos^2 \theta - \tfrac{1}{3}) - \omega^2 a (1 - \cos^2 \theta),$$

or $$g = (1 + \varepsilon)\frac{E}{a^2} - \tfrac{3}{2}\omega^2 a + \left(\tfrac{5}{2}\omega^2 a - \frac{E\varepsilon}{a^2}\right)\cos^2 \theta \ldots (21).$$

At the equator $\theta = \frac{1}{2}\pi$; and if we put G for gravity at the equator, m for the ratio of the centrifugal force to gravity at the equator, we get $\omega^2 a = mG$, and

$$(1 + \tfrac{3}{2}m)\, G = (1 + \varepsilon)\frac{E}{a^2},$$

whence $$E = (1 + \tfrac{3}{2}m - \varepsilon)\, Ga^2 \ldots\ldots\ldots\ldots (22);$$

and (21) becomes $$g = G\left\{1 + (\tfrac{5}{2}m - \varepsilon)\cos^2 \theta\right\} \ldots\ldots\ldots (23).$$

7. Equation (22) gives the mass of the earth by means of the value of G determined by the pendulum. In the preceding investigation, θ is the complement of the corrected latitude; but since θ occurs only in the small terms, and the squares of small quantities have been omitted throughout, we may regard θ as the complement of the true latitude, and therefore replace $\cos \theta$ by the sine of the latitude. In the case of the earth, m is about $\frac{1}{289}$ and ε about $\frac{1}{300}$, and therefore $\frac{5}{2}m - \varepsilon$ is positive. Hence it appears from (23) that the inverse of gravity from the equator to the pole varies as the square of the sine of the latitude, and the ratio which the excess of polar over equatorial gravity bears to the latter, added to the ellipticity, is equal to $\frac{5}{2} \times$ the ratio of the centrifugal force to gravity at the equator.

8. If instead of the equatorial radius a, and equatorial gravity G, we choose to employ the mean radius a_1, and mean gravity G_1, we have only to remark that the mean

value of $\cos^2\theta$, or $\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\cos^2\theta\sin\theta\,d\theta\,d\phi$, is $\frac{1}{3}$, which gives

$$a_1 = a(1 - \tfrac{1}{3}),\quad G_1 = G(1 + \tfrac{5}{6}m - \tfrac{1}{3}\epsilon),$$

which reduces equations (8), (22), and (23) to

$$r = a_1\{1 - \epsilon(\cos^2\theta - \tfrac{1}{3})\},$$

$$E = (1 + \tfrac{2}{3}m)\,G_1 a_1^2,$$

$$g = G_1\{1 + (\tfrac{5}{2}m - \epsilon)(\cos^2\theta - \tfrac{1}{3})\}.$$

9. We get from (19), for the potential at an external point,

$$V = \frac{E}{r} - (\epsilon - \tfrac{1}{2}m)\frac{Ea^2}{r^3}(\cos^2\theta - \tfrac{1}{3})\ \ldots\ldots(24).$$

Now the attraction of the moon on any particle of the earth, and consequently the attraction of the whole earth on the moon, will be very nearly the same as if the moon's mass were collected at her centre of gravity. Let r be the distance between the centres of the earth and moon, θ the moon's north polar distance, P the attraction of the earth on the moon, resolved along the radius vector drawn from the earth's centre, Q the attraction perpendicular to the radius vector, a force which will evidently lie in a plane passing through the earth's axis and the centre of the moon. Then, supposing Q measured positive towards the equator, we have from (4),

$$P = -\frac{dV}{dr},\quad Q = \frac{1}{r}\frac{dV}{d\theta};$$

whence, from (24),

$$\left.\begin{aligned} P &= \frac{E}{r^2} - 3(\epsilon - \tfrac{1}{2}m)\frac{Ea^2}{r^4}(\cos^2\theta - \tfrac{1}{3}) \\ Q &= 2(\epsilon - \tfrac{1}{2}m)\frac{Ea^2}{r^4}\sin\theta\cos\theta \end{aligned}\right\}\ \ldots.\ (25).$$

The moving force arising from the attraction of the earth on the moon is a force passing through the centre of the moon, and having for components MP along the radius vector, and MQ perpendicular to the radius vector, M being the mass of the moon; and an account of the equality of action and reaction, the moving force arising from the attraction of the moon on the earth is equal and opposite to the former. Hence the latter force is equivalent to a moving force MP passing through the earth's centre in the direction of the radius vector of the moon, a force MQ passing through

the earth's centre in a direction perpendicular to the radius vector, and a couple whose moment is MQr tending to turn the earth about an equatorial axis. Since we only want to determine the motion of the moon relatively to the earth, the effect of the moving forces MP, MQ acting on the earth will be fully taken into account by replacing E in equations (25) by $E + M$. If μ be the moment of the couple, we have

$$\mu = 2(\varepsilon - \tfrac{1}{2}m)\frac{MEa^2}{r^3}\sin\theta\cos\theta \;\ldots\ldots\; (26).$$

This formula will of course apply, *mutatis mutandis*, to the moment of the moving force arising from the attraction of the sun.

10. The force expressed by the second term in the value of P, in equations (25), and the force Q, or rather the forces thence obtained by replacing E by $E + M$, are those which produce the only two sensible inequalities in the moon's motion which depend on the oblateness of the earth. We see that they enable us to determine the ellipticity of the earth independently of any hypothesis respecting the distribution of matter in its interior.

The moment μ, and the corresponding moment for the sun, are the forces which produce the phenomena of precession and nutation. In the observed results, the moments of the forces are divided by the moment of inertia of the earth about an equatorial axis. Call this $Ea^2\kappa$; let $M = \frac{1}{n}E$; let b be the annual precession, and f the coefficient in the lunar nutation in obliquity; then we shall have

$$b = \left(A + \frac{B}{n+1}\right)(\varepsilon - \tfrac{1}{2}m)\frac{1}{\kappa}, \quad f = \frac{C}{n+1}(\varepsilon - \tfrac{1}{2}m)\frac{1}{\kappa},*$$

where A, B, C denote certain known quantities. Hence the observed values of b and f will serve to determine the two unknown quantities n, and the ratio of $\varepsilon - \frac{1}{2}m$ to κ. If therefore we suppose ε to be known otherwise, we shall get the numerical value of κ.

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* $\frac{1}{n+1}$ will appear in these equations rather than $\frac{1}{n}$, because, if S be the mass, and $r_{,}$ the distance of the sun, the ratio of $\frac{M}{r^3}$ to $\frac{S}{r_{,}^3}$ is equal to $\frac{1}{n+1}$ multiplied by that of $\frac{E+M}{r^3}$ to $\frac{S}{r_{,}^3}$, and the latter ratio is known by the mean motions of the sun and moon.

11. In determining the mutual attraction of the moon and earth, the attraction of the moon has been supposed the same as if her mass were collected at her centre, which we know would be strictly true if the moon were composed of concentric spherical strata of equal density, and is very nearly true of any mass, however irregular, provided the distance of the attracted body be very great compared with the dimensions of the attracting mass, and the centre be understood to mean the centre of gravity. It will be desirable to estimate the magnitude of the error which is likely to result from this supposition. For this purpose suppose the moon's surface, or at least a surface of equilibrium drawn immediately outside the moon, to be an oblate spheroid of small ellipticity, having its axis of figure coincident with the axis of rotation. Then the equation (24) will apply to the attraction of the moon on the earth, provided we replace E, a, by M, a', where a' is the moon's radius, take θ to denote the angular distance of the radius vector of the earth from the moon's axis, and suppose ε and m to have the values which belong to the moon. Now E is about 80 times as great as M, and a about 4 times as great as a', and therefore Ea^2 is about 1200 times as great as Ma'^2. But m is extremely small in the case of the moon; and there is no reason to think that the value of ε for the moon is large in comparison with its value for the earth, but rather the contrary; and therefore the effect of the moon's oblateness on the relative motions of the centres of the earth and moon must be altogether insignificant, especially when we remember that the coefficients of the two sensible inequalities in the moon's motion depending on the earth's oblateness are only about 8″. It is to be observed that the supposition of a spheroidal figure has only been made for the sake of rendering applicable the equation (24), which had been already obtained, and has nothing to do with the order of magnitude of the terms we are considering.*

Although however the effect of the moon's oblateness, or rather of the possible deviation of her mass from a mass composed of concentric spherical strata, may be neglected in

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* If the expression for V be formed directly, and be expanded according to inverse powers of r, the first term will be $\frac{M}{r}$. The terms involving r^{-2} will disappear if the centre of gravity of the moon be taken for origin, those involving r^{-3} are the terms we are here considering. If the moon's centre of gravity, or rather its projection on the apparent disk, did not coincide with the centre of the disk, it is easy to see the nature of the apparent inequality in the moon's motion which would thence result.

considering the motion of the moon's centre, it does not therefore follow that it ought to be neglected in considering the moon's motion about her own axis. For in the first place, in comparing the effects produced on the moon and on the earth, the moment of the mutual moving force of attraction of the moon and earth is divided by the moment of inertia of the moon, instead of the moment of inertia of the earth, which is much larger; and in the second place, the effect now considered is not mixed up with any other. In fact, it is well known that the circumstance that the moon always presents the same face to us has been accounted for in this manner.

12. In concluding this subject, it may be well to consider the degree of evidence afforded by the figure of the earth in favour of the hypothesis of the earth's original fluidity.

In the first place, it is remarkable that the surface of the earth is so nearly a surface of equilibrium. The elevation of the land above the level of the sea is extremely trifling compared with the breadth of the continents. The surface of the sea must of course necessarily be a surface of equilibrium, but still it is remarkable that the sea is spread so uniformly over the surface of the earth. There is reason to think that the depth of the sea does not exceed a very few miles on the average. Were a roundish solid taken at random, and a quantity of water poured on it, and allowed to settle under the action of the gravitation of the solid, the probability is that the depth of the water would present no sort of uniformity, and would be in some places very great. Nevertheless the circumstance that the surface of the earth is so nearly a surface of equilibrium might be attributed to the constant degradation of the original elevations during the lapse of ages.

In the second place, it is found that the surface is very nearly an oblate spheroid, having for its axis the axis of rotation. That the surface should *on the whole* be protuberant about the equator is nothing remarkable, because even were the matter of which the earth is composed arranged symmetrically about the centre, a surface of equilibrium would still be protuberant in consequence of the centrifugal force; and were matter to accumulate at the equator by degradation, the ellipticity of the surface of equilibrium would be increased by the attraction of this matter. Nevertheless the ellipticity of the earth is much greater than the ellipticity ($\frac{1}{2}m$) due to the centrifugal force alone, and

even greater than the ellipticity which would exist were the earth composed of a sphere touching the surface at the poles, and consisting of concentric spherical strata of equal density and of a spherico-spheroidal shell having the density of the rocks and clay at the surface.* This being the case, the regularity of the surface is no doubt remarkable; and this regularity is accounted for on the hypothesis of original fluidity.

The near coincidence between the numerical values of the ellipticity of the terrestrial spheroid obtained independently from the motion of the moon, from the pendulum, by the aid of Clairaut's theorem, and from direct measures of arcs, affords no additional evidence whatsoever in favour of the hypothesis of original fluidity, being a direct consequence of the law of universal gravitation.†

If the expression for V given by (24) be compared with the expression which would be obtained by direct integration, it may easily be shewn that the axis of rotation is a principal axis, and that the moments of inertia about the

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* It may be proved without difficulty that the value of ε corresponding to this supposition is $\frac{1}{420}$ nearly, if we suppose the density of the shell to be to the mean density as 5 to 11.

† With respect to the argument derived from the motion of the moon, this remark has already been made by Professor O'Brien, who has shewn that if the form of the surface and the law of the variation of gravity be given independently, and if we suppose the earth to consist approximately of spherical strata of equal density, without which it seems impossible to account for the observed regularity of gravity at the surface, then the attraction on the moon follows as a necessary consequence, independently of any theory but that of universal gravitation. (*Tract on the Figure of the Earth.*) If the surface be not assumed to be one of equilibrium, nor even nearly spherical, and if the component of gravity in a direction perpendicular to the surface, as well as the form of the surface, be given independently, it may be shewn that the attraction on an external particle follows, independently of any hypothesis respecting the distribution of matter in the interior of the earth. It may be remarked that if the surface be supposed to differ from a surface of equilibrium by a quantity of the order of the ellipticity, the component of gravity in a direction perpendicular to the surface may be considered equal to the whole force of gravity. Since however, as a matter of fact, the surface *is* a surface of equilibrium, if very trifling irregularities be neglected, it seems better to assume it to be such, and then the law of the variation of gravity, as well as the attraction on the moon, follow from the form of the surface.

It must not here be supposed that these irregularities are actually neglected. Such an omission would ill accord with the accuracy of modern measures. In geodetic operations and pendulum experiments, the direct observations are in fact reduced to the level of the sea, and so rendered comparable with a theory in which it is supposed that the earth's surface is accurately a surface of equilibrium. I have considered this subject in detail in the paper referred to at the beginning of this article, which has since been read before the Cambridge Philosophical Society.

other two principal axes are equal to each other, so that every equatoreal axis is a principal axis. These results would follow as a consequence of the hypothesis of original fluidity. Still it should be remembered that we can only affirm them to be accurate to the degree of accuracy to which we are authorized by measures of arcs and by pendulum experiments to affirm the surface to be an oblate spheroid.

The phenomena of precession and nutation introduce a new element to our consideration, namely the moment of inertia of the earth about an equatoreal axis. The observation of these phenomena enables us to determine the numerical value of the quantity κ, if we suppose ε known otherwise. Now, independently of any hypothesis as to original fluidity, it is probable that the earth consists approximately of spherical strata of equal density. Any material deviation from this arrangement could hardly fail to produce an irregularity in the variation of gravity, and consequently in the form of the surface, since we know that the surface is one of equilibrium. Hence we may assume, when not directly considering the ellipticity, that the density ρ is a function of the distance r from the centre. Now the mean density of the earth as compared with that of water is known from the result of Cavendish's experiment, and the superficial density may be considered equal to that of ordinary rocks, or about $2\frac{1}{2}$ times that of water; and therefore the ratio of the mean to the superficial density may be considered known. Take for simplicity the earth's radius for the unit of length, and let $\rho = \rho_1$ when $r = 1$. From the mean density and the value of κ we know the ratios of the integrals $\int_0^1 \rho r^2 dr$ and $\int_0^1 \rho r^4 dr$ to ρ_1. Now it is probable that ρ increases, at least on the whole, from the surface to the centre. If we assume this to be the case, and restrict ρ to satisfy the conditions of becoming equal to ρ_1 when $r = 1$, and of giving to the two integrals just written their proper numerical values, it is evident that the law of density cannot range within any very wide limits; and speaking very roughly we may say that the density is *determined.*

Now the preceding results will not be sensibly affected by giving to the nearly spherical strata of equal density one form or another, but the form of the surface will be materially affected. The surface in fact might not be spheroidal at all, or if spheroidal, the ellipticity might range between tolerably wide limits. But according to the hypothesis of original fluidity the surface ought to be spheroidal, and the ellipticity

ought to have a certain numerical value depending upon the law of density.

If then there exist a law of density, not in itself improbable *a priori*, which satisfies the required conditions respecting the mean and superficial densities, and which gives to the ellipticity and to the annual precession numerical values nearly agreeing with their observed values, we may regard this law not only as in all probability representing approximately the distribution of matter within the earth, but also as furnishing, by its accordance with observation, a certain degree of evidence in favour of the hypothesis of original fluidity. The law of density usually considered in the theory of the figure of the earth is a law of this kind.

It ought to be observed that the results obtained relative to the attraction of the earth remain just the same whether we suppose the earth to be solid throughout or not; but in founding any argument on the numerical value of κ we are obliged to consider the state of the interior. Thus if the central portions of the earth be, as some suppose, in a state of fusion, the quantity $Ea^2\kappa$ must be taken to mean the moment of inertia of that solid, whatever it may be, which is equivalent to the solid crust together with its fluid or viscous contents. On this supposition it is even conceivable that κ should depend on the period of the disturbing force, so that different numerical values of κ might have to be used in the precession and in the lunar nutation, in which case the mass of the moon deduced from precession and nutation would not be quite correct.

Additional Propositions respecting Attractions.

Although the propositions at the commencement of this paper were given merely for the sake of the applications made of them to the figure of the earth, there are a few additional propositions which are so closely allied to them that they may conveniently be added here.

PROP. VII.* If V be the potential of any mass M_1, and if M_0 be the portion of M_1 contained within a closed surface S,

$$\iint \frac{dV}{dn}\, dS = -\, 4\pi M_0 \ldots\ldots\ldots\ldots (27),$$

* This and Prop. IV. are expressed respectively by equations (7) and (8) in the article by Professor Thomson already referred to (vol. III. p. 203), where a demonstration of a theorem comprehending both founded on the equation

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = -\, 4\pi\rho \ldots\ldots\ldots\ldots\ldots (a)$$

n and dS having the same meaning as in Prop. IV., and the integration being extended to the whole surface S.

Let m' be the mass of an attracting particle situated at the point P' inside S. Through P' draw a right line L, and produce it indefinitely in one direction. This line will in general cut S in one point; but if S be a re-entrant* surface it may be cut by L in three, five, or any odd number of points. About L describe a conical surface containing an infinitely small solid angle a, and let the rest of the notation be as in Prop. IV. In this case the angles $\theta_1, \theta_2, \ldots$ will be alternately obtuse and acute, and we shall have

$$N_1 = -\frac{m'}{r_1^2}\cos(\pi - \theta_1) = \frac{m'}{r_1^2}\cos\theta_1,$$

$$A_1 = ar_1^2 \sec(\pi - \theta_1) = -ar_1^2 \sec\theta_1,$$

and therefore $$N_1 A_1 = -am'.$$

Should there be more than one point of section, the terms N_2A_2, N_3A_3, &c. will destroy each other two and two, as in Prop. IV. Now all angular space around P' may be divided into an infinite number of solid angles such as a, and it is evident that the whole surface S will thus be exhausted. We get therefore

$$\text{limit of } \Sigma NA = -\Sigma am' = -m'\Sigma a;$$

or, since $\Sigma a = 4\pi$, $$\iint N dS = -4\pi m'.$$

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is given. In the present paper a different order of investigation is followed; direct geometrical demonstrations of the equations

$$\iint \frac{dV}{dn}\, dS = 0 \text{ in one case, and } \iint \frac{dV}{dn}\, dS = -4\pi M_0 \text{ in another,}$$

are given in Props. IV. and VII.; and a new proof of the equation (a) is deduced from them in Prop. VIII.

The demonstration here given of the latter of the equations just written is so very similar to Gauss's demonstration of the former, that, trusting to memory, I erroneously stated in the note at p. 195, that the demonstration of Prop. VII. as well as that of Prop. VI. (read IV.) was the same as Gauss's. On referring, however, to Gauss's demonstration of equation (27), I find that it is quite different.

These equations may be obtained as very particular cases of a general theorem originally given by Green (*Essay on Electricity*, p. 12.) It will be sufficient to suppose $U = 1$ in Green's equation, and to observe that $dw = -dn$, and $\delta V = 0$ or $= -4\pi\rho$, if V be taken to denote the potential of the mass whose attraction is considered. This is in fact the method of proof followed in Professor Thomson's paper, which was published before the author had succeeded in meeting with Green's Essay.

* This term is here used, and has been already used in the demonstration of Prop. IV., to denote a closed surface which can be cut by a tangent plane.

The same formula will apply to any other internal particle, and it has been shewn in Prop. IV. that for an external particle $\iint NdS = 0$. Hence, adding together all the results, and taking N now to refer to the attraction of all the particles, both internal and external, we get $\iint NdS = -4\pi M_0$. But $N = \dfrac{dV}{dn}$, which proves the proposition.

Prop. VIII. At an internal point (x, y, z) about which the density is ρ, the potential V satisfies the equation

$$\frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + \frac{d^2 V}{dz^2} = -4\pi\rho \quad \ldots\ldots\ldots \quad (28).$$

Consider the elementary parallelepiped $dx\,dy\,dz$, and apply to it the equation (27). For the face $dy\,dz$ whose abscissa is x, the value of $\displaystyle\iint \frac{dV}{dn}\,dS$ is ultimately $-\dfrac{dV}{dx}\,dy\,dz$, and for the opposite face it is ultimately $+\left(\dfrac{dV}{dx} + \dfrac{d^2 V}{dx^2}\,dx\right) dy\,dz$; and therefore for this pair of faces the value of the integral is ultimately $\dfrac{d^2 V}{dx^2}\,dx\,dy\,dz$. Treating the two other pairs of faces in the same way, we get ultimately for the value of the first member of equation (27),

$$\left(\frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + \frac{d^2 V}{dz^2}\right) dx\,dy\,dz.$$

But the density being ultimately constant, the value of M_0, which is the mass contained within the parallelepiped, is ultimately $\rho\,dx\,dy\,dz$, whence by passing to the limit we obtain equation (28).

The equation which (28) becomes when the polar coordinates r, θ, ϕ are employed in place of rectangular, may readily be obtained by applying equation (27) to the elementary volume $dr \,.\, rd\theta \,.\, r\sin\theta d\phi$, or else it may be derived from (28) by transformation of coordinates. The first member of the transformed equation has already been written down (see equation (10),); the second remains $-4\pi\rho$.

Example of the application of equation (28).—In order to give an example of the practical application of this equation, let us apply it to determine the attraction which a sphere composed of concentric spherical strata of uniform density exerts on an internal particle.

Refer the sphere to polar coordinates originating at the centre. Let ρ be the density, which by hypothesis is a function of r, R the external radius, V the potential of the sphere, which will evidently be a function of r only. For a point within the sphere we get from (28)

$$\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = -4\pi\rho \ldots\ldots\ldots (29).$$

For a point outside the sphere the equation which V has to satisfy is that which would be obtained from (29) by replacing the second member by zero; but we may evidently apply equation (29) to all space provided we regard ρ as equal to zero outside the sphere. Since the first member of (29) is the same thing as $\frac{1}{r}\frac{d^2rV}{dr^2}$, we get

$$V = -\frac{4\pi}{r}\iint \rho r\,dr^2.$$

Now we get by integration by parts,

$$\int(\int\rho r dr)\,dr = r\int\rho r dr - \int\rho r^2 dr,$$

whence $$V = -4\pi\int\rho r dr + \frac{4\pi}{r}\int\rho r^2 dr,$$

where the arbitrary constants are supposed to be included in the signs of integration. Now V vanishes at an infinite distance, and does not become infinite at the centre, and therefore the second integral vanishes when $r = 0$, and the first when $r = \infty$, or, which is the same, when $r = R$, since $\rho = 0$ when $r > R$. We get therefore finally,

$$V = 4\pi\int_r^R \rho r\,dr + \frac{4\pi}{r}\int_0^r \rho r^2 dr.$$

If F be the required force of attraction, we have $F = -\frac{dV}{dr}$; and observing that the two terms arising from the variation of the limits destroy each other, we get

$$F = \frac{4\pi}{r^2}\int_0^r \rho r^2 dr.$$

Now $4\pi\int_0^r \rho r^2 dr$ is the mass contained within a sphere described about the centre with a radius r, and therefore the attraction is the same as if the mass within this sphere were collected at its centre, and the mass outside it were removed.

The attraction of the sphere on an external particle may be considered as a particular case of the preceding, since we may first suppose the sphere to extend beyond the attracted particle, and then make ρ vanish when $r > R$.

Before concluding, one or two more known theorems may be noticed, which admit of being readily proved by the method employed in Prop. v.

PROP. IX. If T be a space which contains none of the attracting matter, the potential V cannot be constant throughout any finite portion of T without having the same constant value throughout the whole of the space T and at its surface. For if possible let V have the constant value A throughout the space T_1, which forms a portion of T, and a greater or less value at the portions of T adjacent to T_1. Let R be a region of T adjacent to T_1 where V is greater than A. By what has been already remarked, V must increase continuously in passing from T_1 into R. Draw a closed surface σ lying partly within T_1 and partly within R, and call the portions lying in T_1 and R, σ_1, σ_2 respectively. Then if ν be a normal to σ, drawn outwards, $\dfrac{dV}{d\nu}$ will be positive throughout σ_1 if σ_1 be drawn sufficiently close to the space T_1 (see Prop. v. and note), and $\dfrac{dV}{d\nu}$ is equal to zero throughout the surface σ_2, since V is constant throughout the space T_1; and therefore $\displaystyle\iint \frac{dV}{d\nu}\, d\sigma$, taken throughout the whole surface σ, will be positive, which is contrary to Prop. IV. Hence V cannot be greater than A in any portion of T adjacent to T_1, and similarly it cannot be less, and therefore V must have the constant value A throughout T, and therefore, on account of the continuity of V, at the surface of T.

Combining this with Prop. v. Cor. 1, we see that if V be constant throughout the whole surface of a space T which contains no attracting matter, it will have the same constant value throughout T; but if V be not constant throughout the whole surface, it cannot be constant throughout any finite portion of T, but only throughout a surface. Such a surface cannot be closed, but must abut upon the surface of T, since otherwise V would be constant within it.

PROP. X. The potential V cannot admit of a maximum or minimum value in the space T.

It appears from the demonstration of Prop. V. that V cannot have a maximum or minimum value at a point, or throughout a line, surface, or space, which is isolated in T. But not even can V have the maximum or minimum value V_1 throughout T_1 if T_1 reach up to the surface S of T; though the term maximum or minimum is not strictly applicable to this case. By Prop. IX. V cannot have the value V_1 throughout a space, and therefore T_1 can only be a surface or a line.

If possible, let V have the maximum value V_1 throughout a line L which reaches up to S. Consider the loci of the points where V has the successive values V_2, V_3..., decreasing by infinitely small steps from V_1. In the immediate neighbourhood of L, these loci will evidently be tube-shaped surfaces, each lying outside the preceding, the first of which will ultimately coincide with L. Let s be an element of L not adjacent to S, nor reaching up to the extremity of L, in case L terminate abruptly. At each extremity of s draw an infinite number of *lines of force*, that is, lines traced from point to point in the direction of the force, and therefore perpendicular to the surfaces of equilibrium. The assemblage of these lines will evidently constitute two surfaces cutting the tubes, and perpendicular to s at its extremities. Call the space contained within the two surfaces and one of the tubes T_2, and apply equation (5) to this space. Since V is a maximum at L, $\dfrac{dV}{dn}$ is negative for the tube surface of T_2, and it vanishes for the other surfaces, as readily follows from equation (4). Hence $\iint \dfrac{dV}{dn}\,dS$, taken throughout the whole surface T_2, is negative, which is contrary to equation (5). Hence V cannot have a maximum value at the line L; and similarly it cannot have a minimum value.

It may be proved in a similar manner that V cannot have a maximum or minimum value V_1 throughout a surface S_1 which reaches up to S. For this purpose it will be sufficient to draw a line of force through a point in S_1, and make it travel round an elementary area σ which forms part of S_1, and to apply equation (5) to the space contained between the surface generated by this line, and the two portions, one on each side of S_1, of a surface of equilibrium corresponding to a value of V very little different from V_1.

It should be observed that the space T considered in this proposition and in the preceding need not be closed: all that

is requisite is that it contain none of the attracting mass. Thus, for instance, T may be the infinite space surrounding an attracting mass or set of masses.

It is to be observed also, that although attractive forces have been spoken of throughout, all that has been proved is equally true of repulsive forces, or of forces partly attractive and partly repulsive. In fact, nothing in the reasoning depends upon the sign of m; and by making m negative we pass to the case of repulsive forces.

Prop. XI. If an isolated particle be in equilibrium under the action of forces varying inversely as the square of the distance, the equilibrium cannot be stable with reference to every possible displacement, nor unstable, but must be stable with reference to some displacements and unstable with reference to others; and therefore the equilibrium of a *free* isolated particle in such circumstances must be unstable.*

For we have seen that V cannot be a maximum or minimum, and therefore either V must be absolutely constant, (as for instance within a uniform spherical shell), in which case the particle may be in equilibrium at any point of the space in which it is situated, or else, if the particle be displaced along any straight line or curve, for some directions of the line or curve V will be increasing and for some decreasing. In the former case the force resolved along a tangent to the particle's path will be directed *from* the position of equilibrium, and will tend to remove the particle still farther from it, while in the latter case the reverse will take place.

NOTES ON HYDRODYNAMICS.

VI.—ON WAVES.

By G. G. Stokes.

The theory of waves has formed the subject of two profound memoirs by MM. Poisson and Cauchy, in which some of the highest resources of analysis are employed, and the results deduced from expressions of great complexity.

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* This theorem was first given by Mr. Earnshaw in his memoir on Molecular Forces read at the Cambridge Philosophical Society, March 18, 1839, (*Trans.* vol. VII.) See also a paper by Professor Thomson in the first series of this *Journal*, vol. IV. p. 223.

This circumstance might naturally lead to the notion that the subject of waves was unapproachable by one who was either unable or unwilling to grapple with mathematical difficulties of a high order. The complexity, however, of the memoirs alluded to arises from the nature of the problem which the authors have thought fit to attack, which is the determination of the motion of a mass of liquid of great depth when a small portion of the surface has been slightly disturbed in a given arbitrary manner. But after all it is not such problems that possess the greatest interest. It is seldom possible to realize in experiment the conditions assumed in theory respecting the initial disturbance. Waves are usually produced either by some sudden disturbing cause, which acts at a particular part of the fluid in a manner too complicated for calculation, or by the wind exciting the surface in a manner which cannot be strictly investigated. What chiefly strikes our attention is the propagation of waves already produced, no matter how: what we feel most desire to investigate is the mechanism and the laws of such propagation. But even here it is not every possible motion that may have been excited that it is either easy or interesting to investigate; there are two classes of waves which appear to be especially worthy of attention.

The first consists of those whose length is very great compared with the depth of the fluid in which they are propagated. To this class belongs the great tidal wave which, originally derived from the oceanic oscillations produced by the disturbing forces of the sun and moon, is propagated along our shores and up our channels. To this class belongs likewise that sort of wave propagated along a canal which Mr. Russell has called a *solitary wave*. As an example of this kind of wave may be mentioned the wave which, when a canal boat is stopped, travels along the canal with a velocity depending, not on the previous velocity of the boat, but merely upon the form and depth of the canal.

The second class consists of those waves which Mr. Russell has called *oscillatory*. To this class belong the waves produced by the action of wind on the surface of water, from the ripples on a pool to the long swell of the Atlantic. By the waves of the sea which are referred to this class must not be understood the surf which breaks on shore, but the waves produced in the open sea, and which, after the breeze that has produced them has subsided, travel along without breaking or undergoing any material change of form. The

theory of oscillatory waves, or at least of what may be regarded as the type of oscillatory waves, is sufficiently simple, although not quite so simple as the theory of long waves.

Theory of Long Waves.

Conceive a long wave to travel along a uniform canal. For the sake of clear ideas, suppose the wave to consist entirely of an elevation. Let k be the greatest height of the surface above the plane of the surface of the fluid at a distance from the wave, where the fluid is consequently sensibly at rest; let λ be the length of the wave, measured suppose from the point where the wave first becomes sensible to where it ceases to be sensible on the opposite side of the ridge; let b be the breadth, and h the depth of the canal if it be rectangular, or quantities comparable with the breadth and depth respectively if the canal be not rectangular. Then the volume of fluid elevated will be comparable with $b\lambda k$. As the wave passes over a given particle, this volume, (not however consisting of the same particles be it observed,) will be transferred from the one side to the other of the particle in question. Consequently if we suppose the horizontal motions of the particles situated in the same vertical plane perpendicular to the length of the canal to be the same, a supposition which cannot possibly give the greatest horizontal motion too great, although previously to investigation it might be supposed to give it too small, the horizontal displacement of any particle will be comparable with $\dfrac{b\lambda k}{bh}$ or $\dfrac{\lambda}{h}k$. Hence if λ be very great compared with h, the horizontal displacements and horizontal velocities will be very great compared with the vertical displacements and vertical velocities. Hence we may neglect the vertical effective force, and therefore regard the fluid as in equilibrium, so far as vertical forces are concerned, so that the pressure at any depth δ below the actual surface will be $g\rho\delta$, g being the force of gravity, and ρ the density of the fluid, the atmospheric pressure being omitted. It is this circumstance that makes the theory of long waves so extremely simple. If the canal be not rectangular, there will be a slight horizontal motion in a direction perpendicular to the length of the canal; but the corresponding effective force may be neglected for the same reason as the vertical effective force, at least if the breadth of the canal be not very great compared with its depth, which is supposed to be the case; and there-

fore the fluid contained between any two infinitely close vertical planes drawn perpendicular to the length of the canal may be considered to be in equilibrium, except in so far as motion in the direction of the length of the canal is concerned. It need hardly be remarked that the investigation which applies to a rectangular canal will apply to an extended sheet of standing fluid, provided the motion be in two dimensions.

Let x be measured horizontally in the direction of the length of the canal; and at the time t draw two planes perpendicular to the axis of x, and passing through points whose abscissæ are x' and $x' + dx'$. Then if η be the elevation of the surface at any point of the horizontal line in which it is cut by the first plane, $\eta + \dfrac{d\eta}{dx'}dx'$ will be the elevation of the surface where it is cut by the second plane. Draw a right line parallel to the axis of x, and cutting the planes in the points P, P'. Then if δ be the depth of the line PP' below the surface of the fluid in equilibrium, the pressures at P, P' will be $g\rho(\delta + \eta)$ and $g\rho\left(\delta + \eta + \dfrac{d\eta}{dx'}dx'\right)$ respectively; and therefore the difference of pressures will be $g\rho\dfrac{d\eta}{dx'}dx'$. About the line PP' describe an infinitely thin cylindrical surface, with its generating lines perpendicular to the planes, and let κ be the area which it cuts from either plane; and consider the motion of fluid which is bounded by the cylindrical surface and the two planes. The difference of the pressures on the two ends is ultimately $g\rho\kappa\dfrac{d\eta}{dx'}dx'$, and the mass being $\rho\kappa\,dx'$, the accelerating force is $g\dfrac{d\eta}{dx'}$. Hence the effective force is the same for all particles situated in the same vertical plane perpendicular to the axis of x; and since the particles are supposed to have no sensible motion before the wave reaches them, it follows that the particles once in a vertical plane perpendicular to the length of the canal remain in such a vertical plane throughout the motion.

Let x be the abscissa of any plane of particles in its position of equilibrium, $x + \xi$ the common abscissa of the same set of particles at the time t, so that ξ and η are functions of x and t. Then equating the effective to the im-

pressed accelerating force, we get

$$\frac{d^2\xi}{dt^2} = -g\,\frac{d\eta}{dx'} \ldots\ldots\ldots\ldots\ldots (1);$$

and we have

$$x' = x + \xi \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Thus far the canal has been supposed to be not necessarily rectangular, nor even uniform, provided that its form and dimensions change very slowly, nor has the motion been supposed to be necessarily very small. If we adopt the latter supposition, and neglect the squares of small quantities, we shall get from (1) and (2)

$$\frac{d^2\xi}{dt^2} = -g\,\frac{d\eta}{dx} \ldots\ldots\ldots\ldots\ldots (3).$$

It remains to form the equation of continuity. Suppose the canal to be uniform and rectangular, and let b be its breadth and h its depth. Consider the portion of fluid contained between two vertical planes whose abscissæ in the position of equilibrium are x and $x + dx$. The volume of this portion is expressed by $bh\,dx$. At the time t the abscissæ of the bounding planes of particles are $x + \xi$ and $x + \xi + \left(1 + \frac{d\xi}{dx}\right) dx$; the depth of the fluid contained between these planes is $h + \eta$; and therefore the expression for the volume is $b\,(h + \eta)\left(1 + \frac{d\xi}{dx}\right) dx$. Equating the two expressions for the volume, dividing by bdx, and neglecting the product of the two small quantities, we get

$$h\,\frac{d\xi}{dx} + \eta = 0^* \ldots\ldots\ldots\ldots\ldots (4).$$

* This equation is in fact a second integral of the ordinary equation of continuity, corrected so as to suit the particular case of motion which is under consideration. For motion in two dimensions the latter equation is

$$\frac{du}{dx} + \frac{dv}{dy} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (a);$$

and denoting by η' the vertical disp acement of any particle, we have

$$u = \frac{d\xi}{dt}, \quad v = \frac{d\eta'}{dt}.$$

Substituting in (a), and integrating with respect to t, we get

$$\frac{d\xi}{dx} + \frac{d\eta'}{dy} = \psi\,(x, y) \ldots\ldots\ldots\ldots\ldots\ldots (b),$$

Eliminating ξ between (3) and (4), we get

$$\frac{d^2\eta}{dt^2} = gh\,\frac{d^2\eta}{dx^2} \dots\dots\dots\dots\dots\dots (5).$$

The complete integral of this equation is

$$\eta = f\{x - \sqrt{(gh)}\,t\} + F\{x + \sqrt{(gh)}\,t\} \dots\dots (6),$$

where f, F denote two arbitrary functions. This integral evidently represents two waves travelling, one in the positive, and the other in the negative direction, with a velocity equal to $\sqrt{(gh)}$, or to that acquired by a heavy body in falling through a space equal to half the depth of the fluid. It may be remarked that the velocity of propagation is independent of the density of the fluid.

It is needless to consider the determination of the arbitrary functions f, F by means of the initial values of η and $\frac{d\eta}{dt}$, supposed to be given, or the reflection of a wave when the canal is stopped by a vertical barrier, since these investigations are precisely the same as in the case of sound, or in that of a vibrating string. The only thing peculiar to the present problem consists in the determination of the motion of the individual particles.

It is evident that the particles move in vertical planes parallel to the length of the canal. Consider an elementary column of fluid contained between two such planes infinitely close to each other, and two vertical planes, also infinitely close to each other, perpendicular to the length of the canal. By what has been already shewn, this column of fluid will remain throughout the motion a vertical column on a rectangular base; and since there can be no vertical motion at the bottom of the canal, it is evident that the vertical displacements of the several particles in the column will be proportional to their heights above the base. Hence it will be sufficient to determine the motion of a particle at the surface; when the motion of a particle at a given depth will be found

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$\psi(x, y)$ denoting an arbitrary function of x, y, that is, a quantity which may vary from one particle to another, but is independent of the time. To determine ψ we must observe that when any particle is not involved in the wave $\eta' = 0$, and ξ does not vary in passing from one particle to another, and therefore $\psi(x, y) = 0$. Integrating equation (b) with respect to y from $y = 0$ to $y = h + \eta$, observing that ξ is independent of y, and that the limits of η' are 0 and η, and neglecting $\eta\,\frac{d\xi}{dx}$, which is a small quantity of the second order, we get the equation in the text.

by diminishing in a given ratio the vertical displacement of the superficial particle immediately above it, without altering the horizontal displacement.

The motion of a particle at the surface is defined by the values of η and ξ. The former is given by (6), where the functions f, F are now supposed known, and the latter will be obtained from (4) by integration. Consider the case in which a single wave consisting of an elevation is travelling in the positive direction: let λ be the length of the wave, and suppose the origin taken at the posterior extremity of the wave in the position it occupies when $t = 0$; then we may suppress the second function in (6), and we shall have $f(x) = 0$ from $x = -\infty$ to $x = 0$, and from $x = \lambda$ to $x = +\infty$, and $f(x)$ will be positive from $x = 0$ to $x = \lambda$. Let

$$c = \sqrt{(gh)} \dots\dots\dots\dots\dots\dots (7),$$

so that c is the velocity of propagation, and let the position of equilibrium of a particle be considered to be that which it occupies before the wave reaches it, so that ξ vanishes for $x = +\infty$. Then we have from (4) and (6)

$$\xi = -\frac{1}{h}\int_{\infty}^{x} \eta \, dx = \frac{1}{h}\int_{x}^{\infty} f(x - ct)\, dx \dots\dots (8).$$

Consider a particle situated in front of the wave when $t = 0$, so that $x > \lambda$. Since $f(x) = 0$ when $x > \lambda$, we shall have $f(x - ct) = 0$, until $ct = x - \lambda$. Consequently from (6) and (8) there will be no motion until $t = \dfrac{x - \lambda}{c}$, when the motion will commence. Suppose now that a very small portion only of the wave, of length s, has passed over the particle considered. Then $x - ct = \lambda - s$; and we have from (6) and (8)

$$\eta = f(\lambda - s),\ \xi = \frac{1}{h}\int_{-\infty}^{s} f(\lambda - s)\, ds = \frac{1}{h}\int_{0}^{s} f(\lambda - s)\, ds:$$

for since $f(x)$ vanishes when $x > \lambda$, we may replace the limits $-\infty$ and s by 0 and s. Since $\int_0^s f(\lambda - s)\, ds$ is equal to s multiplied by the mean value of $f(\lambda - s)$ from 0 to s, and this mean value is comparable with $f(\lambda - s)$, it follows that ξ is at first very small compared with η. Hence the particle begins to move vertically; and since η is positive the motion takes place upwards. As the wave advances, ξ becomes sensible, and goes on increasing positively. Hence the particle moves forwards as well as upwards. When the

ridge of the wave reaches the particle, η is a maximum; the upward motion ceases, but it follows from (8) that ξ is then increasing most rapidly, so that the horizontal velocity is a maximum. As the wave still proceeds, η begins to decrease, and ξ to increase less rapidly. Hence the particle begins to descend, and at the same time its onward velocity is checked. As the wave leaves the particle, it may be shewn just as before that the final motion takes place vertically downwards. When the wave has passed, $\eta = 0$, so that the particle is at the same height from the bottom as at first; but ξ is a positive constant, equal to $\frac{1}{h}\int_0^\lambda f(x)\,dx$, or to $\frac{1}{bh}\int_0^\lambda bf(x)\,dx$, that is, to the volume elevated divided by the area of the section of the canal. Hence the particle is finally deposited in advance of its initial position by the space just named.

If the wave consists of a single depression, instead of a single elevation, everything is the same as before, except that the particle is depressed and then raised to its original height, in place of being first raised and then depressed, and that it is moved backwards, or in a direction contrary to that of propagation, instead of being moved forwards.

These results of theory with reference to the motions of the individual particles may be compared with Mr. Russell's experiments described at page 342 of his second report on Waves.*

In the preceding investigation the canal has been supposed rectangular. A very trifling modification, however, of the preceding process will enable us to find the velocity of propagation in a uniform canal, the section of which is of any arbitrary contour. In fact, the dynamical equation (3) will remain the same as before; the equation of continuity alone will have to be altered. Let A be the area of a section of the canal, b the breadth at the surface of the fluid; and consider the mass of fluid contained between two vertical planes whose abscissæ in the position of equilibrium are x and $x + dx$, and which therefore has for its volume Adx. At the time t, the distance between the bounding planes of particles is $\left(1 + \frac{d\xi}{dx}\right) dx$, and the area of a section of the fluid is $A + b\eta$ nearly, so that the volume is

$$(A + b\eta)\left(1 + \frac{d\xi}{dx}\right) dx, \text{ or } \left(A + b\eta + A\frac{d\xi}{dx}\right) dx$$

* Report of the 14th meeting of the British Association. Mr. Russell's first report is contained in the Report of the 7th meeting.

nearly. Equating the two expressions for the volume, we get

$$A\frac{d\xi}{dx} + b\eta = 0 .$$

Comparing this equation with (4), we see that it is only necessary to write $\frac{A}{b}$ for h; so that if c be the velocity of propagation,

$$c = \sqrt{\left(\frac{gA}{b}\right)} \text{ .... } (9).$$

The formula (9) of course includes (7) as a particular case. The last mentioned was given long ago by Lagrange;* the more comprehensive formula (9) was first given by Prof. Kelland,† though at the same time or rather earlier it was discovered independently by Green,‡ in the particular case of a triangular canal. These formulæ agree very well with experiment when the height of the waves is small, which has been supposed to be the case in the previous investigation, as may be seen from Mr. Russell's reports. A table containing a comparison of theory and experiment in the case of a triangular canal is given in Green's paper. In this table the mean error is only about $\frac{1}{60}$th of the whole velocity.

As the object of this note is merely to give the simplest cases of wave motion, the reader is referred to Mr. Airy's treatise on tides and waves for the effect produced by a slow variation in the dimensions of the canal on the length and height of the wave,‖ as well as for the effect of the finite height of the wave on the velocity of propagation. With

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* Berlin Memoirs, 1786, p. 192. In this memoir Lagrange has obtained the velocity of propagation by very simple reasoning. Laplace had a little earlier (*Mém. de l'Académie* for 1776, p. 542) given the expression (see equation (29) of this note) for the velocity of propagation of oscillatory waves, which when h is very small compared with λ reduces itself to Lagrange's formula, but had made an unwarrantable extension of the application of the formula. In the *Mécanique Analytique* Lagrange has obtained analytically the expression (7) for the velocity of propagation when the depth is small, whether the motion take place in two or three dimensions, by assuming the result of an investigation relating to sound.

For a full account of the various theoretical investigations in the theory of waves, which had been made at the date of publication, as well as for a number of interesting experiments, the reader is referred to a work by the brothers Weber, entitled "Wellenlehre auf Experimente gegründet," *Leipsig*, 1825.

† Transactions of the Royal Society of Edinburgh, Vol. xiv. pp. 524, 530.

‡ Transactions of the Cambridge Philosophical Society, Vol. vii. p. 87.

‖ Encyclopædia Metropolitana. Art. 260 of the treatise.

respect to the latter subject, however, it must be observed that in the case of a solitary wave artificially excited in a canal it does not appear to be sufficient to regard the wave as infinitely long when we are investigating the correction for the height; it appears to be necessary to take account of the finite length, as well as finite height of the wave.

Theory of Oscillatory Waves.

In the preceding investigation, the general equations of hydrodynamics have not been employed, but the results have been obtained by referring directly to first principles. It will now be convenient to employ the general equations. The problem which it is here proposed to consider is the following.

The surface of a mass of fluid of great depth is agitated by a series of waves, which are such that the motion takes place in two dimensions. The motion is supposed to be small, and the squares of small quantities are to be neglected. The motion of each particle being periodic, and expressed, so far as the time is concerned, by a circular function of given period, it is required to determine all the circumstance of the motion of the fluid. The case in which the depth is finite and uniform will be considered afterwards.

It must be observed that the supposition of the periodicity of the motion is not, like the hypothesis of parallel sections, a mere arbitrary hypothesis introduced in addition to our general equations, which, whether we can manage them or not, are sufficient for the complete determination of the motion in any given case. On the contrary, it will be justified by the result, by enabling us to satisfy all the necessary equations; so that it is used merely to define, and select from the general class of possible motions, that particular kind of motion which we please to contemplate.

Let the vertical plane of motion be taken for the plane of xy. Let x be measured horizontally, and y vertically upwards from the mean surface of the fluid. If a, b be the coordinates of any particle in its mean position, the coordinates of the same particle at the time t will be $a + \int u dt$, $b + \int v dt$, respectively. Since the squares of small quantities are omitted, it is immaterial whether we conceive u and v to be expressed in terms of a, b, t, or in terms of x, y, t; and, on the latter supposition, we may consider x and y as constant in the integration with respect to t. Since the variable terms in the expressions for the coordinates are supposed to con-

tain t under the form $\sin nt$ or $\cos nt$, the same must be the case with u and v. We may therefore assume

$$u = u_1 \sin nt + u_2 \cos nt,\ v = v_1 \sin nt + v_2 \cos nt,$$

where u_1, u_2, v_1, v_2 are functions of x and y without t. Substituting these values of u and v in the general equations of motion, neglecting the squares of small quantities, and observing that the only impressed force acting on the fluid is that of gravity, we get

$$\frac{1}{\rho}\frac{dp}{dx} = -nu_1 \cos nt + nu_2 \sin nt,$$

$$\frac{1}{\rho}\frac{dp}{dy} = -g - nv_1 \cos nt + nv_2 \sin nt \ldots . \ (10),$$

and the equation of continuity becomes

$$\left(\frac{du_1}{dx} + \frac{dv_1}{dy}\right) \sin nt + \left(\frac{du_2}{dx} + \frac{dv_2}{dy}\right) \cos nt = 0 \ . \ . (11).$$

Eliminating p by differentiation from the two equations (10), we get

$$\left(\frac{du_1}{dy} - \frac{dv_1}{dx}\right) \cos nt - \left(\frac{du_2}{dy} - \frac{dv_2}{dx}\right) \sin nt = 0 \ . \ . (12);$$

and in order that this equation may be satisfied, we must have separately

$$\frac{du_1}{dy} - \frac{dv_1}{dx} = 0,\ \frac{du_2}{dy} - \frac{dv_2}{dx} = 0 \ldots\ldots . (13).$$

The first of these equations requires that $u_1 dx + v_1 dy$ be an exact differential $d\phi_1$, and is satisfied merely by this supposition. Similarly the second requires that $u_2 dx + v_2 dy$ be an exact differential $d\phi_2$. The functions ϕ_1, ϕ_2 may be supposed not to contain t, provided that in integrating equations (10) we express explicitly an arbitrary function of t instead of an arbitrary constant. In order to satisfy (11) we must equate separately to zero the coefficients of $\sin nt$ and $\cos nt$. Expressing u_1, v_1, u_2, v_2 in terms of ϕ_1, ϕ_2 in the resulting equations, we get

$$\frac{d^2\phi_1}{dx^2} + \frac{d^2\phi_1}{dy^2} = 0 \ \ldots\ldots\ldots\ldots (14),$$

with a similar equation for ϕ_2. Integrating the value of dp given by (10), we get

$$\frac{p}{\rho} = -gy - n\phi_1 \cos nt + n\phi_2 \sin nt + \psi(t) \ . \ . (15).$$

It remains to form the equation of condition which has to be satisfied at the free surface. If we suppose the atmospheric pressure not to be included in p, we shall have $p = 0$ at the free surface; and we must have at the same time (Note II.)

$$\frac{dp}{dt} + u\frac{dp}{dx} + v\frac{dp}{dy} = 0 \text{ .... } (16).$$

The second term in this equation is of the second order, and in the third we may put for $\frac{dp}{dy}$ its approximate value $-g\rho$. Consequently at the free surface, which is defined by the equation

$$gy + n\phi_1 \cos nt - n\phi_2 \sin nt - \psi(t) = 0 \text{....} (17),$$

we must have

$$n^2\phi_1 \sin nt + n^2\phi_2 \cos nt + \psi'(t) - g\left(\frac{d\phi_1}{dy}\sin nt + \frac{d\phi_2}{dy}\cos nt\right) = 0$$
$$\text{........ } (18):$$

and we have the further condition that the motion shall vanish at the infinite depth. Since the value of y given by (17) is a small quantity of the first order, it will be sufficient after differentiation to put $y = 0$ in (18).

Equations (18), (14), and the corresponding equation for ϕ_2 shew that the functions ϕ_1, ϕ_2 are independent of each other; and (15), (17) shew that the pressure at any point and the ordinate of the free surface are composed of the sums of the parts due to these two functions respectively. Consequently we may temporarily suppress one of the functions ϕ_2, which may be easily restored in the end by writing $t + \frac{\pi}{2n}$ for t, and changing the arbitrary constants.

Equation (14) may be satisfied in the most general way by an infinite number of particular solutions of the form $A\varepsilon^{m'x+my}$, where any one of the three constants A, m', m may be positive or negative, real or imaginary, and m', m are connected by the equation $m'^2 + m^2 = 0$.* Now m' cannot be wholly real, nor partly real and partly imaginary, since in that case the corresponding particular solution would become infinite either for $x = -\infty$ or for $x = +\infty$, whereas the fluid is supposed to extend indefinitely in the direction of x, and the expressions

* See Poisson, *Traité de Mécanique*, tom. II. p. 347, or *Théorie de la Chaleur*, chap. V.

for the velocity, &c., must not become infinite for any point of space occupied by the fluid. Hence m' must be wholly imaginary, and therefore m wholly real. Moreover m must be positive, since otherwise the expression considered would become infinite for $y = -\infty$. The equation connecting m and m' gives $m' = \pm m\sqrt{(-1)}$. Uniting in one the two corresponding solutions with their different arbitrary constants, we have for the most general particular solution which we are at liberty to take $(A\varepsilon^{mx\sqrt{(-1)}} + B\varepsilon^{-mx\sqrt{(-1)}})\,\varepsilon^{my}$, which becomes, on replacing the imaginary exponentials by circular functions, and changing the arbitrary constants,

$$(A \sin mx + B \cos mx)\,\varepsilon^{my}.$$

Hence we must have

$$\phi_1 = \Sigma\,(A \sin mx + B \cos mx)\,\varepsilon^{my} \;\ldots\ldots (19)$$

the sign Σ denoting that we may take any number of positive values of m with the corresponding values of A and B.

Substituting now in (18), supposed to be deprived of the function ϕ_2, the value of ϕ_1 given by (19), and putting $y = 0$ after differentiation, we have

$$\sin nt\,\Sigma\,(n^2 - mg)\,(A \sin mx + B \cos mx) + \psi'(t) = 0.$$

Since no two terms such as $A \sin mx$ or $B \cos mx$ can destroy each other, or unite with the term $\psi'(t)$, we must have separately $\psi'(t) = 0$, and

$$n^2 - mg = 0 \;\ldots\ldots\ldots\ldots\ldots (20).$$

The former of these equations gives $\psi(t) = k$, where k is a constant; but (17) shews that the mean value of the ordinate y of the free surface is $\dfrac{k}{g}$, inasmuch as ϕ_1 and ϕ_2 consist of circular functions so far as x is concerned, and therefore we must have $k = 0$, since we have supposed the origin of coordinates to be situated in the mean surface of the fluid. The latter equation restricts (19) to one particular value of m.

To obtain ϕ_2 it will be sufficient to take the expression for ϕ_1 with new arbitrary constants. If we put ϕ for

$$\phi_1 \sin nt + \phi_2 \cos nt, \quad \text{so that } \phi = \int(udx + vdy),$$

we see that ϕ consists of four terms, each consisting of the product of an arbitrary constant, a sine or cosine of nt, a sine or cosine of mx and of the same function ε^{my} of y. By replacing the products of the circular functions by sines or

cosines of sums or differences, and changing the arbitrary constants, we shall get four terms multiplied by arbitrary constants, and involving sines and cosines of $mx - nt$ and of $mx + nt$. The terms involving $mx - nt$ will represent a disturbance travelling in the positive direction, and those involving $mx + nt$ a disturbance travelling in the negative direction. If we wish to consider only the disturbance which travels in the positive direction, we must suppress the terms involving $mx + nt$, and we shall then have got only two terms left, involving respectively $\sin(mx - nt)$ and $\cos(mx - nt)$. One of these terms, whichever we please, may be got rid of by altering the origin of x; and we may therefore take

$$\phi = A \sin(mx - nt)\varepsilon^{my} \ldots\ldots\ldots (21);$$

and ϕ determines, by its partial differential coefficients with respect to x and y, the horizontal and vertical components of the velocity at any point. We have from (21), and the definitions of ϕ_1, ϕ_2,

$$\phi_1 = -A \cos mx \,.\, \varepsilon^{my}, \quad \phi_2 = A \sin mx \,.\, \varepsilon^{my}.$$

Substituting in (15) and (17), putting $\psi(t) = 0$, and replacing y by 0 in the second and third terms of (17), we get

$$\frac{p}{\rho} = g(-y) + nA \cos(mx - nt)\,\varepsilon^{my} \ldots\ldots (22),$$

which gives the pressure at any point, and

$$y = \frac{nA}{g} \cos(mx - nt) \ldots\ldots\ldots (23)^*,$$

which gives the equation to the free surface at any instant.

If λ be the length of a wave, T its period, c the velocity of propagation, we have $m = \frac{2\pi}{\lambda}$, $n = \frac{2\pi}{T}$, $n = cm$; and therefore from (20)

$$c = \sqrt{\left(\frac{g\lambda}{2\pi}\right)} \ldots\ldots\ldots\ldots (24).$$

Hence the velocity of propagation varies directly, and the period of the wave inversely, as the square root of the wave's length. Equation (23) shews that a section of the surface at any instant is the curve of sines.

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* Equations (22), (23) may be got at once from the equations

$$\frac{p}{\rho} = -gy - \frac{d\phi}{dt}, \quad 0 = gy + \frac{d\phi}{dt}.$$

It may be remarked that in consequence of the form of ϕ equation (18) is satisfied, not merely for $y = 0$, but for any value of y; and therefore (16) is satisfied, not merely at the free surface, but throughout the mass. Hence the pressure experienced by a given particle is constant throughout the motion. This is not true when the depth is finite, as may be seen from the value of ϕ adapted to that case, which will be given presently; but it may be shewn to be true when the depth is infinite, whether the motion take place in two, or three dimensions, and whether it be regular or irregular, provided it be small, and be such that $udx + vdy + wdz$ is an exact differential.

It will be interesting to determine the motions of the individual particles. Let $x + \xi$, $y + \eta$ be the coordinates of the particle whose mean position has for coordinates x, y. Then we have

$$\frac{d\xi}{dt} = u = \frac{d\phi}{dx}, \quad \frac{d\eta}{dt} = v = \frac{d\phi}{dy};$$

and in the values of u, v we may take x, y to denote the actual coordinates of any particle or their mean values indifferently, on account of the smallness of the motion. Hence we get from (21), after differentiation and integration,

$$\xi = -\frac{mA}{n}\sin(mx - nt)\,\varepsilon^{my}, \quad \eta = \frac{mA}{n}\cos(mx - nt)\varepsilon^{my} \quad ..(25).$$

Hence the particles describe circles about their mean places, with a uniform angular motion. Since η is a maximum at the same time with y in (23), and $\frac{d\xi}{dt}$ is then positive, any particle is in its highest position when the crest of the wave is passing over it, and is then moving horizontally forwards, that is, in the direction of propagation. Similarly any particle is in its lowest position when the middle of the trough is passing over it, and it is then moving horizontally backwards. The radius of the circle described is equal to $\frac{mA}{n}\varepsilon^{my}$, and it therefore decreases in geometric progression as the depth of the particle considered increases in arithmetic. The rate of decrease is such that at a depth equal to λ the displacement is to the displacement at the surface as $\varepsilon^{-2\pi}$ to 1, or as 1 to 535 nearly.

If the depth of the fluid be finite, the preceding solution may of course be applied without sensible error, provided

ε^{my} be insensible for a negative value of y equal to the depth of the fluid. This will be equally true whether the bottom be regular or irregular, provided that in the latter case we consider the depth to be represented by the least actual depth.

Let us now suppose the depth of the fluid finite and uniform. Let h be the mean depth of the fluid, that is, its depth as unaffected by the waves. It will be convenient to measure y from the bottom rather than from the mean surface. Consequently we must put $y = h$, instead of $y = 0$, in the values of ϕ_1, ϕ_2, and their differential coefficients, in (17) and (18). The only essential change in the equations of condition of the problem is, that the condition that the motion shall vanish at an infinite depth is replaced by the condition that the fluid shall not penetrate into, or separate from the bottom, a condition which is expressed by the equation

$$\frac{d\phi}{dy} = 0 \text{ when } y = 0 \quad \dots\dots\dots\dots \quad (26).$$

Everything is in the same as in the preceding investigation till we come to the selection of a particular integral of (14). As before, y must appear in an exponential, and x under a circular function; but both exponentials must now be retained. Hence the only particular solution which we are at liberty to take is of the form

$$A\varepsilon^{my} \cos mx + B\varepsilon^{my} \sin mx + C\varepsilon^{-my} \cos mx + D\varepsilon^{-my} \sin mx,$$

or, which is the same thing, the coefficients only being altered,

$$(\varepsilon^{my} + \varepsilon^{-my})(A \cos mx + B \sin mx)$$
$$+ (\varepsilon^{my} - \varepsilon^{-my})(C \cos mx + D \sin mx).$$

Now (26) must be satisfied by ϕ_1 and ϕ_2 separately. Substituting then in this equation the value of ϕ_1 which is made up of an infinite number of particular values of the above form, we see that we must have for each value of m in particular $C = 0$, $D = 0$; so that

$$\phi_1 = \Sigma\, (\varepsilon^{my} + \varepsilon^{-my})(A \cos mx + B \sin mx)\,.$$

Substituting in equation (18), in which ϕ_2 is supposed to be suppressed, and y put equal to h after differentiation, we get

$$n^2 (\varepsilon^{mh} + \varepsilon^{-mh}) - mg\, (\varepsilon^{mh} - \varepsilon^{-mh}) = 0 \quad \dots\dots \quad (27),$$

and $\psi'(t) = 0$, which gives $\psi(t) = k$. The equation (17) shews that this constant k must be equal to h, which is the mean value of y at the surface. It is easy to prove that

equation (27), in which m is regarded as the unknown quantity, has one and but one positive root. For, putting $mh = \mu$, and denoting by ν the function of μ defined by the equation

$$\nu\,(\varepsilon^{\mu} + \varepsilon^{-\mu}) = \mu\,(\varepsilon^{\mu} - \varepsilon^{-\mu}) \ldots\ldots\ldots (28),$$

we get by taking logarithms and diffentiating

$$\frac{1}{\nu}\frac{d\nu}{d\mu} = \frac{1}{\mu} + \frac{\varepsilon^{\mu} + \varepsilon^{-\mu}}{\varepsilon^{\mu} - \varepsilon^{-\mu}} - \frac{\varepsilon^{\mu} - \varepsilon^{-\mu}}{\varepsilon^{\mu} + \varepsilon^{-\mu}}.$$

Now the right-hand member of this equation is evidently positive when μ is positive; and since ν is also positive, as appears from (28), it follows that $\frac{d\nu}{d\mu}$ is positive; and therefore μ and ν increase together. Now (28) shews that ν passes from 0 to ∞ as μ passes from 0 to ∞, and therefore for one and but one positive value of μ, ν is equal to the given quantity $\frac{n^2 h}{g}$, which proves the theorem enunciated. Hence as before the most general value of ϕ corresponds to two series of waves, of determinate length, which are propagated, one in the positive, and the other in the negative direction. If c be the velocity of propagation, we get from (27), since $n = cm = c\,.\,\frac{2\pi}{\lambda}$,

$$c = \left\{\frac{g\lambda}{2\pi}\,\frac{1 - \varepsilon^{-\frac{4\pi h}{\lambda}}}{1 + \varepsilon^{-\frac{4\pi h}{\lambda}}}\right\}^{\frac{1}{2}} \ldots\ldots\ldots\ldots (29).$$

If we consider only the series which is propagated in the positive direction, we may take for the same reason as before

$$\phi = A\,(\varepsilon^{my} + \varepsilon^{-my})\sin(mx - nt) \ldots\ldots (30);$$

which gives

$$\frac{p}{\rho} = g\,(h - y) + nA\,(\varepsilon^{my} + \varepsilon^{my})\cos(mx - nt) \,.\,.\, (31),$$

and for the equation to the free surface

$$g\,(y - h) = nA\,(\varepsilon^{mh} + \varepsilon^{-mh})\cos(mx - nt) \,.\,.\, (32).$$

Equations (21), (22), (23) may be got from (30), (31), (32) by writing $y + h$ for y, $A\varepsilon^{-mh}$ for A, and then making h infinite. When λ is very small compared with h, the formula (29) reduces itself to (24): when on the contrary λ is very

great it reduces itself to (7). It should be observed however that this mode of proving equation (7) for very long waves supposes a section of the surface of the fluid to be the curve of sines, whereas the equation has been already obtained independently of any such restriction.

The motion of the individual particles may be determined, just as before, from (30). We get

$$\xi = -\frac{mA}{n}(\varepsilon^{my} + \varepsilon^{my}) \sin(mx - nt),$$

$$\eta = \frac{mA}{n}(\varepsilon^{my} - \varepsilon^{my}) \cos(mx - nt) \ldots\ldots (33).$$

Hence the particles describe elliptic orbits, the major axes of which are horizontal, and the motion in the ellipses is the same as in the case of a body describing an ellipse under the action of a force tending to the centre. The ratio of the minor to the major axis is that of $1 - \varepsilon^{-2my}$ to $1 + \varepsilon^{-2my}$, which diminishes from the surface downwards, and vanishes at the bottom, where the ellipses pass into right lines.

The ratio of the horizontal displacement at the depth $h - y$ to that at the surface is equal to the ratio of $\varepsilon^{my} + \varepsilon^{-my}$ to $\varepsilon^{mh} + \varepsilon^{-mh}$. The ratio of the vertical displacements is that of $\varepsilon^{my} - \varepsilon^{-my}$ to $\varepsilon^{mh} - \varepsilon^{-mh}$. The former of these ratios is greater, and the latter less than that of $\varepsilon^{-m(h-y)}$ to 1. Hence, for a given length of wave, the horizontal displacements decrease less, and the vertical displacements more rapidly from the surface downwards when the depth of the fluid is finite, than when it is infinitely great.

In a paper "On the Theory of Oscillatory Waves"* I have considered these waves as mathematically defined by the character of uniform propagation in a mass of fluid otherwise at rest, so that the waves are such as could be propagated into a portion of fluid which had no previous motion, or excited in such a portion by means of forces applied to the surface. It follows from the latter character, by virtue of the theorem proved in Note IV, that $udx + vdy$ is an exact differential. This definition is equally applicable whether the motion be or be not very small; but in the present note I have supposed the species of wave considered to be defined by the character of periodicity, which perhaps forms a somewhat simpler definition when the motion is small. In the paper just mentioned I have proceeded to a second approxi-

* Cambridge Philosophical Transactions, vol. VIII. p. 441.

mation, and in the particular case of an infinite depth to a third approximation. The most interesting result, perhaps, of the second approximation is, that the ridges are steeper and narrower than the troughs, a character of these waves which must have struck everybody who has been in the habit of watching the waves of the sea, or even the ripples on a pool or canal. It appears also from the second approximation that in addition to their oscillatory motion the particles have a progressive motion in the direction of propagation, which decreases rapidly from the surface downwards. The factor expressing the rate of decrease in the case in which the fluid is very deep is ε^{-2my}, y being the depth of the particle considered below the surface. The velocity of propagation is the same as to a first approximation, as might have been seen *a priori*, because changing the sign of the coefficient denoted by A in equations (21) and (30) comes to the same thing as shifting the origin of x through a space equal to $\frac{1}{2}\lambda$, which does not alter the physical circumstances of the motion; so that the expression for the velocity of propagation cannot contain any odd powers of A. The third approximation in the case of an infinite depth gives an increase in the velocity of propagation depending upon the height of the waves. The velocity is found to be equal to $c_0\left(1+\frac{2\pi^2a^2}{\lambda^2}\right)$, c_0 being the velocity given by (24), and a the height of the waves above the mean surface, or rather the coefficient of the first term in the equation to the surface.

A comparison of theory and observation with regard to the velocity of propagation of waves of this last sort may be seen at pages 271 and 274 of Mr. Russell's second report. The following table gives a comparison between theory and experiment in the case of some observations made by Capt. Stanley, R.N. The observations were communicated to the British Association at its late meeting at Swansea. The numbers given in the first two columns are taken from the *Athenæum* of Sept. 2, 1841, to which, or to the forthcoming volume of reports of the Association, the reader is referred for details.

In the following table

A is the length of a wave, in fathoms;

B is the velocity of propagation deduced from the observations, expressed in knots per hour;

C is the velocity given by the formula (24), the observations being no doubt made in deep water;

D is the difference between the numbers given in columns B and C.

In calculating the numbers in table C, I have taken $g = 32.2$ feet, and expressed the velocity in knots of 1000 fathoms or 6000 feet*.

A	B	C	D
55	27.0	24.7	2.3
43	24.5	21.8	2.7
50	25.0	23.5	0.5
35 to 40	22.1	20.4	1.7
33	22.1	19.1	3.0
57	26.2	25.1	1.1
35	22.0	19.7	2.3

The mean of the numbers in column D is 1.94, nearly, which is about the one-eleventh of the mean of those in column C. The quantity 1.94 appears to be less than the most probable error of any one observation, judging by the details of the experiments; but as all the errors lie in one direction, it is probable that the formula (24) gives a velocity a little too small to agree with observations under the circumstances of the experiments. The height of the waves from crest to trough is given in experiments No. 1, 2, 3, 6, 7, by numbers of feet ranging from 17 to 22. I have calculated the theoretical correction for the velocity of propagation depending upon the height of the waves, and found it to be .5 or .6 of a knot, by which the numbers in column C ought to be increased. But on the other hand, according to theory, the particles at the surface have a progressive motion of twice that amount; so that if the ship's velocity, as measured by the log-line, were the velocity relatively to the surface of the water, her velocity would be under-estimated to the amount of 1 or 1.2 knot, which would have to be added to the numbers in column B, or which is the same subtracted from those in column C, in order to compare theory and experiment; so that on the whole .5 or .6 would have to be subtracted from the numbers in column C. But on account of the depth to which the ship sinks in the sea, and the rapid decrease of the factor ε^{-2my} from the surface downwards, the

* I have taken a knot to be 1000 fathoms rather than 2040 yards, because the former value appears to have been used in calculating the numbers in column B.

correction 1 or 1.2 for the "heave of the sea"* would be too great; and therefore, on the whole, the numbers in column C may be allowed to stand. If the numbers given in Capt. Stanley's column, headed "Speed of Ship" already contain some such correction, the numbers in column C must be increased, and therefore those in column D diminished, by .5 or .6.

It has been supposed in the theoretical investigation that the surface of the fluid was subject to a uniform pressure. But in the experiments the wind was blowing strong enough to propel the ship at the rate of from 5 to 7.8 knots an hour. There is nothing improbable in the supposition that the wind might have slightly increased the velocity of propagation of the waves.

There is one other instance of wave motion which may be noticed before we conclude. Suppose that two series of oscillatory waves, of equal magnitude, are propagated in opposite directions. The value of ϕ which belongs to the compound motion will be

$$(\varepsilon^{my} + \varepsilon^{-my})\,(A \cos [mx - nt] + A \cos [mx + nt + a]),$$

the squares of small quantities being neglected, as throughout this note. Since

$$\cos(mx - nt) + \cos(mx + nt + a) = 2 \cos\left(mx + \frac{a}{2}\right) \cos\left(nt + \frac{a}{2}\right),$$

we get by writing $\frac{1}{2}A$ for A, and altering the origins of x and t, so as to get rid of a,

$$\phi = A\,(\varepsilon^{my} + \varepsilon^{-my}) \cos mx \,.\, \cos nt. \,\ldots\ldots\ldots (34).$$

This is in fact one of the elementary forms already considered, from which two series of progressive oscillatory waves were derived by merely replacing products of sines and cosines by sums and differences. Any one of these four elementary forms corresponds to the same kind of motion as any other, since any two may be derived from each other by merely altering the origins of x and t; and therefore it will be sufficient to consider that which has just been written. We get from (34)

$$u = -\,mA\,(\varepsilon^{my} + \varepsilon^{-my}) \sin mx \cos nt,$$

$$v = mA\,(\varepsilon^{my} - \varepsilon^{-my}) \cos mx \cos nt. \,\ldots\ldots (35).$$

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* I have been told by a naval friend that an allowance for the "heave of the sea" is sometimes actually made. As well as I recollect, this allowance might have been about 10 knots a day for waves of the magnitude of those here considered.

We have also for the equation to the free surface

$$y - h = \frac{nA}{g}(\varepsilon^{my} + \varepsilon^{-my}) \cos mx \sin nt \dots . (36).$$

Equations (35) shew that for an infinite series of planes for which $mx = 0, = \pm \pi, = \pm 2\pi$, &c., *i. e.* $x = 0, = \pm \frac{1}{2}\lambda = \pm \lambda$, &c. there is no horizontal motion, whatever be the value of t; and for planes midway between these the motion is entirely horizontal. When $t = 0$, (36) shews that the surface is horizontal; the particles are then moving with their greatest velocity. As t increases, the surface becomes elevated (A being supposed positive) from $x = 0$ to $x = \frac{1}{4}\lambda$, and depressed from $x = \frac{1}{4}\lambda$ to $x = \frac{1}{2}\lambda$, which sufficiently defines the form of the whole, since the planes whose equations are $x = 0$, $x = \frac{1}{2}\lambda$, are planes of symmetry. When $nt = \frac{1}{2}\pi$, the elevation or depression is the greatest; the whole fluid is then for an instant at rest, after which the direction of motion of each particle is reversed. When nt becomes equal to π, the surface again becomes horizontal; but the direction of each particle's motion is just the reverse of what it was at first, the magnitude of the velocity being the same. The previous motion of the fluid is now repeated in a reverse direction, those portions of the surface which were elevated becoming depressed, and *vice versâ*. When $nt = 2\pi$, everything is the same as at first. Equations (35) shew that each particle moves backwards and forwards in a right line.

This sort of wave, or rather oscillation, may be seen formed more or less perfectly when a series of progressive oscillatory waves is incident perpendicularly on a vertical wall. By means of this kind of wave the reader may if he pleases make experiments for himself on the velocity of propagation of small oscillatory waves, without trouble or expense. It will be sufficient to pour some water into a rectangular box, and, first allowing the water to come to rest, to set it in motion by tilting the box, turning it round one edge. The oscillations may be conveniently counted by watching the bright spot on the wall or ceiling occasioned by the light of the sun reflected from the surface of the water, care being taken not to have the motion too great. The time of oscillation from rest to rest is half the period of a wave, and length of the interior edge parallel to the plane of motion is half the length of a wave; and therefore the velocity of propagation will be got by dividing the length of the edge by the time of oscillation. This velocity is then to be compared with the formula (29).

ON THE PROBLEM TO DETERMINE IN MAGNITUDE, POSITION, AND FIGURE, THE SURFACE OF THE SECOND ORDER WHICH PASSES THROUGH NINE GIVEN POINTS.

By Richard Townsend.

In the solution about to be offered of this problem, the author claims nothing more than to have applied and brought together the results of other enquirers, in a manner which, considering the nature of the problem itself, he conceives will not be altogether devoid of interest. The principal steps are all obvious and immediate applications of the principles contained in the valuable paper of Dr. Hesse "On the Construction of the Surface of the Second Order which passes through Nine given Points," of which Mr. Cayley rendered an important service to the readers of this *Journal* by furnishing the abstract contained in the preceding number. The only other step which presents any difficulty has been long ago solved by M. Chasles; and the remainder of the solution is such as could not but readily present itself to any enquirer at all familiarly acquainted with the most ordinary properties of surfaces of the second order.

The problem, after a certain stage in the progress of its solution, naturally, and indeed necessarily, *resolves itself into two*, according as it will appear whether the surface passing through the nine points *will be of the central or of the non-central class:* from that out the two cases, being in fact *two distinct problems*, will require two different methods of treatment, and the whole will be admitted to be completely solved, if we can determine in the first case, *the centre of the surface, its species, and its three semi-axes in direction and magnitude;* and in the second case, *the vertex of the surface, its axis, and any plane section perpendicular to its axis. The cone*, should the surface be of any degenerate form, being included in the former case, and *the three cylinders* including *every case of two planes* being contained in the latter: thus exhausting every possible variety, even *imaginary* surfaces, though such from the very nature of the enquiry are of course incapable of ever occurring in the question before us.

Leaving all *particular* cases to be indicated under their respective heads, the following solution, as it first presented itself to the author on reading the abstract referred to, has direct reference only to the *two general* cases, and may conveniently be subdivided in both into three different parts or steps, as follows:—

Firstly. In the case of a *central surface* of any species.

1stly. *From the nine given points, to determine (by application of Dr. Hesse's principles) the position of the finitely distant centre, and the directions of three conjugate diameters of the surface.*

2ndly. *From any three conjugate diameters given in direction, to find (either by repetition of the same process or independently by means of any three of the nine points) the magnitudes of the three semi-diameters, and the species of the surface.*

3rdly. *From any three conjugate semi-diameters given in magnitude and direction, to determine in direction and magnitude the three semi-axes of the surface.*

Secondly. In the case of a *paraboloid* of any species.

1stly. *From the nine given points, to determine the direction of the infinitely distant centre, and the positions of two conjugate diametral planes of the surface.*

2ndly. *From these to determime the position of the axis, the directions of two conjugate diametral planes intersecting in the axis, and the vertex of the surface.*

3rdly. *And from these with any two of the nine points, to determine the section of the surface by any assumed plane perpendicular to its axis.*

All these different steps, with the exception of the third in the first case, may be solved without going beyond the limits of merely elementary geometry, by means of Dr. Hesse's "*Construction for determining the polar plane of any assumed point with respect to the surface.*" This we proceed to shew in what follows, solving as we advance the corresponding steps of the two different cases simultaneously, or at least as much together as possible.

To solve the first step.

Taking arbitrarily *a point at infinity* in any direction, find by the method in that paper, which not only holds but becomes much simplified in its application for *infinitely distant* points, its polar plane with respect to the surface, which will be *a diametral plane*, whatever be the nature of the surface. Taking then *another* point at infinity *in any direction parallel to the plane so found*, find again its polar plane with respect to the surface, which will be a *second* diametral plane *conjugate to the former*, and that whether the surface be central or non-central. And, taking finally *a third* point at infinity *in the particular direction parallel to the line of intersection of the two planes already determined*, find, lastly, its polar plane with respect to the surface, which will be *the third*

diametral plane conjugate to the two former in the case of any of the central surfaces, but which will be *at an infinite distance in the case of any of the paraboloids.*

Here then at this stage, without going any further, we see whether the particular surface passing through the nine given points will be a central surface or a paraboloid; and at the same time we obtain in the former case the position of the centre and the directions of three conjugate diameters, and in the latter case the direction of the axis and the positions of two conjugate diametral planes. Q. E. D.

There is one case however in which the preceding construction fails, viz. when the *second* diametral plane determined in the foregoing manner comes out to be *parallel to the first;* but then that is the very case in which the particular species of the surface may be determined in the most complete manner at this early stage of the process. For the circumstance supposed, indicating the existence both of an *infinitely distant diameter* and at the same time of *a diametral plane parallel to its own system of ordinates*, shews that the surface must be either *an hyperbolic paraboloid* or *a parabolic cylinder*, the *elliptic paraboloid* not admitting of the latter property, while it is the only other surface which admits of the former: which of the two species the surface will actually be will be at once determined by taking the *third* infinitely distant point in some direction *not* parallel to the two planes; for if then its polar plane remain still parallel to the two former, the surface will be a parabolic cylinder, but if not it will be an hyperbolic paraboloid.

What therefore we may be already supposed to know of the surface, is whether it will be, 1$^{\text{stly}}$, a *central surface*, including the *cone*, in which case we know the position of its centre with the directions of three conjugate diameters; 2$^{\text{ndly}}$, a *paraboloid*, including the *elliptic* and *hyperbolic cylinders* with *two intersecting planes*, in which case we know the positions of two conjugate diametral planes with the direction of its axis; or, 3$^{\text{rdly}}$, a *parabolic cylinder* including *two parallel planes*, in which case we know the common direction of all the diametral planes. In the second step, to which we now proceed, the actual species of the surface in the first and second cases will be definitely determined.

To solve the second step.

In the first case. Taking arbitrarily any three new points XYZ at any three *finite* distances *on the known directions of the three conjugate diameters*, and determining by the same method their three polar planes with respect to the surface,

the three planes so found *will be parallel respectively to the three corresponding diametral planes*, and will intersect the diameters themselves in three points $X'Y'Z'$ respectively, such that the three rectangles under the distances of the three pairs of corresponding points X and X', Y and Y', Z and Z' from the centre, will be equal respectively to the squares of the three corresponding semi-diameters of the surface. Hence *the three new polar planes give us at once the required magnitudes of the three conjugate semi-diameters.*

And at the same time they give us also *the particular species of the central surface* which passes through the nine points. Since for every two of the three pairs of points X and X', Y and Y', Z and Z', *which lie at the same side of the centre*, the corresponding semi-diameter will be *real*, while for every pair *which lie at opposite sides* of that point it will on the contrary be *imaginary;* should one of the three polar planes *pass through the centre*, that is, if one of the three points $X'Y'Z'$ *actually coincide with that point*, so will also the other two, and *the surface will be a cone.*

In the second case. Taking two new *infinitely distant* points in directions *parallel to the two finitely distant planes already determined*, and both *perpendicular to the diameter of the surface* in which those planes intersect, and determining their polar planes by a repetition of the same process as before, we get two new conjugate diametral planes *intersecting in the axis of the surface;* and then taking arbitrarily a third new point at any *finite* distance *on* the axis so found, and bisecting the portion of the axis intercepted between this last point and its polar plane determined in the same manner as all the preceding, *the point of bisection so found will be the vertex of the surface.*

Should the *second* pair of diametral planes in the construction just given come out not only to be parallel but *actually to coincide with the first pair*, or should the polar plane of the *finitely distant* point on the axis come out to be situated not at a finite but *at an infinite distance*, the surface *will be a cylinder*, *elliptic* or *hyperbolic*, according to circumstances; but otherwise it will be *a paraboloid in its general form*, which will be of the elliptic or of the hyperbolic species as the case may be: in both cases the *precise species*, which as yet remains undecided, will be fully determined in the third step at the same time with the *actual magnitude* and *exact form* of the surface.

Having now solved the first and second steps for both cases, we ought next, in the proper order of our subject,

to get on immediately to the third part. But here we may pause for a moment with advantage, as some remarks may be made which will render the practical determination of the polar planes for the *finitely distant* points in the preceding investigation much less troublesome in the particular cases considered, than if arrived at from the construction afforded by the certainly elegant but at the same time practically complicated method of Dr. Hesse.

Firstly, then, we may observe that the *directions* of the required planes being known in all the four cases from other considerations, it will be sufficient in each case to find *a single point through which the plane must pass:* and this remark is peculiarly applicable to Dr. Hesse's general construction, inasmuch as in it the polar plane is always determined *by finding successively three different points through which it must pass, and that too by the mere repetition three times in succession of precisely the same series of operations performed with three analogous but different systems of right lines and planes.*

But again: as the construction for determining the polar planes by that method *from the nine given points*, though reduced in the cases above to one third of the usual process, is still practically sufficiently complicated, it may not be altogether amiss to shew that in the case of the *central* surfaces, *the magnitudes of the three conjugate semi-diameters with the species of the surface may be obtained independently from the known directions of the three diameters and from any three of the nine points,* by a method which consists in finding the three polar planes of *a particular system* of three points assumed on the three directions, and which leads to a construction for these required planes much simpler in many cases than Dr. Hesse's, even when reduced in the manner already mentioned.

Let O be the centre of the surface, ABC three of the given points through which it must pass, and XYZ the three points in which the plane of those points intersects the known directions of the three conjugate diameters. The three polar planes of these three particular points XYZ with respect to the surface, will as before be parallel respectively to the three conjugate central planes OYZ, OZX, OXY, and will meet their diameters OX, OY, OZ, in three points $X'Y'Z'$, from which, from their *distances* and *directions* from the centre O, we shall have at once, as in the former case, the squares of the three semi-diameters both in *magnitude* and *sign*, and therefore also the *species* of the surface including the cone: hence, if we can determine those three polar planes, we shall be in complete possession of all we require.

Let UVW be the three points where the three right lines YZ, ZX, XY, in which the plane of ABC intersects the opposite diametral planes respectively pierce infinity; GHK the three points in which the sides BC, CA, AB of the triangle ABC meet any one of those right lines, YZ for example; LMN the three other points on those sides harmonic conjugates to GHK with respect to their pairs of extremities B and C, C and A, A and B respectively; then will the five polar planes of the five points $GHKYZ$ pass respectively through the five points $LMNVW$, and intersect all in the same right line parallel to OX, which will pierce the plane ABC in a certain point P, which we have now to determine.

The anharmonic ratio of any faisceau of four planes being always equal to that of their four poles, we have in the plane of ABC the anharmonic ratios of the two pencils of four lines $P.LMNV$ and $P.LMNW$, equal respectively to those of the two systems of four points $GHKY$ and $GHKZ$; but of those ratios the two latter are given, therefore so are also the two former. Hence the position of P may be considered as fully determined, to find it being reduced to an obvious and easy problem of elementary geometry, viz. *to find a point in a plane, such that, joining it with five given points, any two required systems of four of the joining lines shall have given anharmonic relations;** a problem which presents no sort of difficulty in consequence of the two pencils, whichever they may be, having necessarily three of their four rays common.

By repetition of the same process to the other two right lines ZX and XY in which the plane of ABC intersects the other two conjugate diametral planes, we get two other points Q and R similarly determined in the plane ABC. *Three right lines drawn through the three points P, Q, and R, thus obtained parallel to the corresponding diameters OX, OY, and OZ respectively, will them pass all through a point D, which will be the pole of the plane ABC with respect to the surface; and the three planes QRD, RPD, and PQD formed by those lines two and two will be the three polar planes of the points X, Y, and Z respectively.*

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* That there can exist but one such point is manifest, were there no other way of shewing it, from observing that it is the fourth point of intersection of the two conics, loci of the two systems of points in the plane for which each anharmonic ratio independently of the other is separately constant; the other three intersections being the three points common to the rays of both pencils. The problem however admits of a very simple elementary solution, of which the most simple case is that which occurs above, that namely in which the two unique points which are not common to the two pencils are both at an infinite distanc.

Thirdly. We may observe that the construction just given for determining the polar planes of the three finitely distant points in the case of the central surfaces ***holds also for determining the polar plane of the single finitely distant point on the axis of the surface in the case of the paraboloids,*** with a very slight modification of which the only effect is but to render it simpler: for, provided the point X and the right lines XY and XZ, in which the plane of the three points ABC intersects the axis, and the two conjugate diametral planes which intersect in the axis be at a finite distance, the construction as above given, is manifestly independent of whether the other two points Y, Z with the third right line YZ be at a finite or at an infinite distance, and therefore holds for the particular point X, as well for the paraboloids as for the central surfaces.

And, lastly, we may observe that the two analogous problems in plane geometry, viz. ***Given the positions of two conjugate diameters of a conic, and two points on the curve, to determine the magnitudes of the two semi-diameters and the species of the conic;*** and ***Given the position of the axis of a parabola and two points on the curve, to determine the position of the vertex,*** though in comparison with the above they present no sort of difficulty, will be wanting in the next step and therefore may be noticed here. Hence, in the first case, O as before being the given centre, A and B the two points, and X and Y the two points in which their right line intersects the given direction of the two conjugate diameters, let H and K be the two points of harmonic section of AB conjugate to X and Y respectively; then drawing through those points parallels HX', KY' to OY and OX respectively, the two points X' and Y' in which they intersect their own diameters will give at once, by their distances and directions from the centre O, the squares of the two semi-diameters both in magnitude and sign, and the species of the curve. And in the second case, X being the point in which the line AB intersects the given position of the axis, let H be the point of harmonic section of AB conjugate to X; then, letting fall a perpendicular HX' from H on the axis, and bisecting the interval XX' between its foot X' and the point X, the point of bisection will be the vertex of the curve.

To solve the third step.

Case 1. If it have appeared from the preceding parts that the surface will be ***either of the two hyperboloids*** in their general form, this part (like the analogous problem in plane geometry) presents very little difficulty: for having the three central

sections of the surface in the three conjugate diametral planes, of which in either case two must be hyperbolas, both of them always real, and the third an ellipse, real or imaginary as the case may be, we have at once *in the asymptotes of the two hyperbolas, four sides of the asymptotic cone of the surface,* and *in the position and species of the ellipse with the direction of the diameter conjugate to its plane,* we have *those of the parallel elliptic sections of the cone with the right line locus of their centres.* Hence *we have the asymptotic cone,* and hence to find the axes of the surface we have therefore *but to find those of the cone,* which, though not determinable by *elementary* geometry, may be found in a multitude of different ways, of which perhaps the simplest is by means of *the four points of its intersection with the conic focal to any one of its plane sections arbitrarily assumed, for which the three pairs of planes passing through those four points and through the vertex of the cone will intersect two and two in the three axes required,* and from the once known *directions* of the three semi-axes of the surface their *magnitudes* may be obtained in a variety of ways.

Case 2. If it have appeared (which is the case of far greatest interest, but at the same time the only one which presents any considerable amount of difficulty) that the surface will be *an ellipsoid;* then the above construction *failing* in consequence of the asymptotic cone being imaginary, that which has been substituted for it by the illustrious Chasles affords perhaps the most beautiful and ingenious application of *the principle of contingent relations* to be met with in the whole range of modern geometry: this we subjoin as briefly as possible, partly because it seems not to be as generally known as its elegance and importance deserve, and partly also to shew, from a theorem of the late Professor MacCullagh, that the very same construction by which the *directions* of the three semi-axes of the surface were determined by M. Chasles, *gives at the same time their magnitudes also,* a circumstance which seems not to have been observed by the latter geometer when he first gave the solution to which we refer.

M. Chasles not being able of course to employ the asymptotic cone *of the surface itself,* determined in place of it by a highly original and ingenious construction the *two* asymptotic cones *to a particular pair of hyperboloids of different species,* confocal with the ellipsoid,* and having got them, the three pairs

* For which reason M. Chasles' construction fails in its turn when applied to the case of either of the two hyperboloids.

of planes passing through their four sides of intersection, gave him by their intersections, two and two, the directions of the three axes he sought. This whole construction for the directions and (as we shall see) for the magnitudes also of the semi-axes of the surface is well worth the attention of every geometer in this branch of his subject, both for its own sake and also for that of the principles on which it depends, which are briefly as follows.

Any two *reciprocally convertible* systems of three confocal surfaces of the second order being taken arbitrarily, *the intersection, normals, and tangent planes of each being the centre axes and principal planes of the other*; we have always the following very general and symmetrical properties reciprocally true of both systems, and that independently of the circumstance whether the different quantities to which they refer are imaginary or real.*

1. *The three rectangular semi-axes of any one surface of either system are always equal to the three coincident semi-axes of the three surfaces of the other system, which coincide in direction with the normal to that particular surface of the first system.*

2. *The two rectangular semi-axes of the central section of any one surface of either system, conjugate to the centre of the other system, are always equal to the two coincident semi-axes of the three focal conics of the latter system, which coincide in direction with the normal to that particular surface of the former system.*

3. *The series of cones asymptotic to the series of surfaces confocal with either system, always envelopes the series of surfaces confocal with the other system; and three particular asymptotis cones of each series pass always through the three focal conics of the other series.*

4. *The three planes drawn through the centre of either system, parallel to the three principal planes of the other, always make equal angles with the four bifocal lines drawn from the centre of the latter through any two of the three focal conics of the first system, and always intercept on each of those four lines three portions equal to the three coincident semi-axes of the first system, which are perpendicular to the plane of its third focal conic, and therefore equal to the three rectangular semi-axes of the corresponding surface of the other system to which they, themselves are respectively perpendicular.*

Of these principles, the three first, which give the directions of the axes in the problem before us, were originally

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* For demonstrations of these different properties, see Professor MacCullagh's paper "On Surfaces of the Second Order," in the *Proceedings of the Royal Irish Academy*, vol. II.

due to M. Chasles himself; and the fourth, which gives at the same time their magnitudes also, was subsequently due to Professor MacCullagh. From the three given conjugate semi-diameters of the ellipsoid, M. Chasles *determined the two real focal conics, the ellipse and hyperbola, of the system of three surfaces reciprocal to the ellipsoid which have their centre at the extremity of any one of the three given semi-diameters* in the following manner.

Having let fall a perpendicular from the extremity of the assumed semi-diameter upon the plane of the other two, and having determined in the usual manner the two semi-axes of the central section in that plane in magnitude and direction, he drew through that perpendicular two rectangular planes parallel to the directions of those semi-axes, and he measured off on the perpendicular in both directions, from the extremity of the semi-diameter, two portions equal to their magnitudes; the extremities of the portions thus taken off he then made the vertices and foci of an ellipse and hyperbola focal to each other in the two rectangular planes, the *ellipse* being in the plane parallel to the *lesser* axis of the central section, and the *hyperbola* in that parallel to the *greater:* these (by properties 1 and 2) are *the two real* focal conics of the system of surfaces reciprocal to the ellipsoid.

Two cones from the given centre subtending the two conics so found, are (by property 3) *coaxal with the ellipsoid, and intersect in the four real bifocal lines of the reciprocal system of surfaces.* Hence *the three pairs of planes passing through those four right lines, intersect two and two in the directions of the three axes required.*

Having thus obtained the *directions* of the three semi-axes, M. Chasles had recourse to one or two of the variety of ways in which *the squares of their magnitudes* may be obtained from the known magnitudes and the ascertained directions with respect to the axes of the three given conjugate semi-diameters: but this now is no longer necessary, as Professor MacCullagh's theorem (property 4) gives immediately *their actual lengths* in the following manner.

The three planes drawn through the extremity of the assumed semi-diameter perpendicular to the above determined directions, will intersect on each of the aforesaid bifocal lines three portions equal to the lengths of the semi-axes of the ellipsoid to which they are respectively perpendicular.

Thus we have the complete solution of this highly interesting case.

CASE 3. If it have appeared (which is a case of little or no interest as compared with either of the two preceding) that

the surface will be *either of the two paraboloids* in their general form, then may this remaining step also, like those which preceded it, be solved without going beyond the limits of elementary geometry: for having the vertex and axis of the surface with a pair of conjugate diametral planes intersecting in the axis, we can determine by means of any two of the nine given points, *the section of the surface by a plane perpendicular to the axis at any assumed distance from the vertex,* which of course determines the surface itself in magnitude, species, position, and form: for since, when we have the vertex, axis, and a point of any parabola, we have at once, by simple proposition, the point in which it intersects any plane perpendicular to its axis, hence in the case before us *we have of the section in question the centre, the directions of a pair of conjugate diameters, and two points on the curve,* from which in the manner already stated with prospective reference to this and the following case, we obtain, first the magnitudes of the two semi-diameters with the species of the conic, and then, from the two conjugate semi-diameters known in magnitude and direction, we get in direction and magnitude the two semi-axes of the curve.

Case 4. If it have appeared (which is the case of far the least interest) that the surface will be *of any degenerate variety, either a cone or any of the three cylinders* in their general or in any particular form, then is this remaining step of the problem evidently one of simple elementary geometry, viz. *to describe, as the case may be, either a conic passing through five or a parabola passing through four given points,* or, which is the same thing, *from the centre the directions of two conjugate diameters and two points to determine the curve in the former case, and from the axis and two points to determine it in the latter.*

Thus the problem is completely solved in all its different cases.

The Lemma on which Dr. Hesse so ingeniously founded the whole process in his elegant solution, viz. "*The polar planes of a fixed point with respect to a system of surfaces of the second order which pass through seven given points will all turn round a fixed point,*" was given by Mr. Salmon in his Examination Papers in October, 1843, and as his proof of the theorem is remarkably simple, while on the contrary it was only stated by Mr. Cayley in his Abstract in the last number of the *Journal,* we give it in conclusion as supplying an interesting step with which all may perhaps not be familiar. Let $S=0$ be the equation of the system of surfaces passing through the seven given points referred for symmetry to any system of *quadri-*

linear coordinates $\alpha\beta\gamma\delta$, and therefore *homogeneous* in the variables, and containing *two independent* parameters, and let $A = 0$, $B = 0$, $C = 0$, be those of *any three particular surfaces* of the system, then will $S = 0$ be in general of the form $aA + bB + cC = 0$, a, b, and c being any three arbitrary constants, let also $\alpha'\beta'\gamma'\delta'$ be the coordinates of the fixed point, and let D stand for the symbol

$$\left(\alpha' \frac{d}{d\alpha} + \beta' \frac{d}{d\beta} + \gamma' \frac{d}{d\gamma} + \delta' \frac{d}{d\delta}\right),$$

then* will the equation of the polar plane of $\alpha'\beta'\gamma'\delta'$ with respect to $S = 0$ be $DS = 0$, a plane which therefore, whatever be the values of a, b, and c, manifestly passes always through the point of intersection of the three fixed planes $DA = 0$, $DB = 0$, $DC = 0$.

I have endeavoured without success to find a purely geometrical proof either of this Lemma or of its reciprocal, that *if a surface of the second order touch seven planes, the locus of its centre, or more generally of the pole of any fixed plane, will be a plane*, were such obtained, the whole investigation from the beginning would be entirely geometrical.

Trinity College, Dublin, March 26th, 1849.

ON THE TRIPLE TANGENT PLANES TO A SURFACE OF THE THIRD ORDER.

By the Rev. George Salmon.

The following are intended as notes supplementary to Mr. Cayley's paper on the same subject, which appeared in the *Journal* (vol. IV. p. 118).

I.

Mr. Hart has proposed the following notation for the 27 right lines on a surface of the third degree. The right lines are denoted by letters of three alphabets, as follows:

$$A_1,\ B_1,\ C_1;\quad A_2,\ B_2,\ C_2;\quad A_3,\ B_3,\ C_3;$$
$$a_1\ \ b_1,\ c_1;\quad a_2,\ b_2,\ c_2;\quad a_3,\ b_3,\ c_3;$$
$$\alpha_1,\ \beta_1,\ \gamma_1;\quad \alpha_2,\ \beta_2,\ \gamma_2;\quad \alpha_3,\ \beta_3,\ \gamma_3.$$

Letters of the same alphabet denote lines which meet if

* Salmon's *Conic Sections*, Art. 287.

either the letters or the suffixes be the same; for example, $A_1A_2A_3$ denote lines on the same plane as also do $A_1B_1C_1$.

Letters of different alphabets denote lines which meet, according to the following table:

$a_1\ b_2\ c_3$ A_1 $\alpha_1\ \beta_2\ \gamma_3$	$b_1\ c_2\ a_3$ B_1 $\beta_1\ \gamma_2\ \alpha_3$	$c_1\ a_2\ b_3$ C_1 $\gamma_1\ \alpha_2\ \beta_3$
$c_2\ a_3\ b_1$ A_2 $\beta_3\ \gamma_1\ \alpha_2$	$a_2\ b_3\ c_1$ B_2 $\gamma_3\ \alpha_1\ \beta_2$	$b_2\ c_3\ a_1$ C_2 $\alpha_3\ \beta_1\ \gamma_2$
$b_3\ c_1\ a_2$ A_3 $\gamma_2\ \alpha_3\ \beta_1$	$c_3\ a_1\ b_2$ B_3 $\alpha_2\ \beta_3\ \gamma_1$	$a_3\ b_1\ c_2$ C_3 $\beta_2\ \gamma_3\ \alpha_1$

The letter in the centre of each square denotes a line which meets each vertical pair of lines. Thus then the five planes through A_1 are $A_1A_2A_3$, $A_1B_1C_1$, $A_1a_1\alpha_1$, $A_1b_2\beta_2$, $A_1c_3\gamma_3$.

The manner in which the table is formed is sufficiently obvious.

II.

At p. 129 Mr. Cayley gives the following theorem:—"Considering two lines in the same treble tangent plane, the remaining treble tangent planes through these two lines respectively are homologous systems." Mr. Hart has given a geometrical demonstration of this theorem, which admits of being stated here even without the help of the diagram of the section of the system by a treble tangent plane, by which he illustrates it. Take for example the lines A_1C_1, and it is required to prove that the systems

$$\left.\begin{matrix} \gamma_3\ \alpha_1\ \beta_2\ A_2 \\ A_1 \\ c_3\ a_1\ b_2\ A_3 \end{matrix}\right\},\qquad \left.\begin{matrix} a_2\ b_3\ c_1\ C_2 \\ C_1 \\ \alpha_2\ \beta_3\ \gamma_1\ C_3 \end{matrix}\right\}$$

are homologous systems.

Now if we take the points where the first system is met by C_3 and the second by A_3, it will be seen that both belong to the new system

$$\begin{matrix} \gamma_3,\ \alpha_1,\ \beta_2,\ A_3C_3, \\ B_2 \\ a_2,\ b_3,\ c_1,\ A_3C_3, \end{matrix}$$

the last plane denoting the plane passing through the line B_2 and the point A_3C_3.

Or again, the points where the first system is met by C_2 and the second by A_2, lie on four planes passing through B_3.

From Mr. Cayley's theorem may be derived the following. "Take any one of the points of contact of each of the five planes through any line on the surface, these form a system homologous to that of the remaining five points of contact."

Thus if we take the points where the line B_2 meets the two systems already employed, we find that the points where B_2 meets $\gamma_3\, \alpha_1\, \beta_2\, A_2$ form a system homologous to that of the points where B_2 meets $a_2\, b_3\, c_1\, C_2$; but the central square of the table shews that these are points of contact of four planes through B_2.

III.

This latter theorem is a particular case of the following more general one, "All the planes drawn through any right line on a surface of the third degree touch the surface in pairs of points which form a system *in involution,* or such that the anharmonic ratio of any four is equal to that of the corresponding four."

It would follow hence, by the known property of a system in involution, that two planes can be drawn to touch the surface in two coincident points; or such that the sections made by them shall be a right line and a conic touching that right line.

Let the points of contact of these planes be π, π', and it is plain that π and π' must be the points where the right line meets the parabolic curve on the surface.

It would follow also from the properties of a system in involution, that if any plane through the right line touch the surface in the points a, a', then the line is cut harmonically in the points $a\pi a'\pi'$: and moreover that the rectangle $Oa.Oa'$ is constant; where O is the finite point of contact of the plane through the right line whose other point of contact is at an infinite distance.

The proof of these theorems is easy. Let the equation of the surface be

$$\kappa xyz + \omega\,(x^2 + y^2 + Axy + \&c.) = 0.$$

Then the points of contact of any plane through $z\omega$, $(z = \mu\omega)$ are given by the equation

$$x^2 + (A + \mu\kappa)\,xy + y^2 = 0.$$

The roots of this must be of the form

$$x + \lambda y = 0\,; \quad x + \frac{1}{\lambda}\, y = 0.$$

The points π, π', correspond to the values of μ given by the equations

$$A + \mu\kappa = \pm\, 2\,;$$

and it will at once be seen that the four planes

$$\lambda x + y, \quad x + \lambda y, \quad x + y, \quad x - y,$$

form an harmonic pencil.

IV.

We have just seen that every right line on a surface of the third degree meets the parabolic curve in only two points: but since this curve is the intersection of the surface with one of the 4th degree, it follows that the 27 right lines are each of them double tangents to the parabolic curve. This theorem may be extended to the following: "If a right line be wholly contained on any surface, it will touch the parabolic curve of that surface in $n - 2$ points." Let us put

$$a = \frac{d^2u}{dx^2}, \quad b = \frac{d^2u}{dy^2}, \quad c = \frac{d^2u}{dz^2}, \quad d = \frac{d^2u}{d\omega^2}:$$

$$l = \frac{d^2u}{dydz}, \quad m = \frac{d^2u}{dzdx}, \quad n = \frac{d^2u}{dxdy},$$

$$p = \frac{d^2u}{dxd\omega}, \quad q = \frac{d^2u}{dyd\omega}, \quad r = \frac{d^2u}{dzd\omega}.$$

Then the parabolic curve is given by the equation

$$l^2p^2 + m^2q^2 + n^2r^2 - 2mnqr - 2nlrp - 2lmpq + abcd + 2alqr$$
$$+ 2bmpr + 2cnpq + 2dlmn - abr^2 - bcp^2 - caq^2 - adl^2 - bdm^2 - cdn^2 = 0.$$

Now if the equation of the surface be of the form

$$x\phi + y\psi = 0,$$

we have for the points where the parabolic curve meets the line xy,

$$c = 0, \quad d = 0, \quad r = 0,$$

and the equation above reduces to

$$(lp - mq)^2 = 0, \quad \text{or to} \quad \left(\frac{d\phi}{dz}\,\frac{d\psi}{d\omega} - \frac{d\phi}{d\omega}\,\frac{d\psi}{dz}\right)^2 = 0.$$

V.

At the conclusion of his paper (p. 132) Mr. Cayley remarks that, "the preceding theory is very materially modified if

the surface have one or more conical points." I wish now to shew that it may still be stated in general terms that "*every* surface of the third degree (not being a ruled surface) contains 27 right lines and has 45 treble tangent planes;" but that a line or plane through a conical point must be counted as two; through two conical points as four, and a plane through three conical points as eight. By similar modifications the general enunciation can be made to include the case where the surface has one or more double points at which the tangent cone breaks up into two planes. The following enumeration contains the results of a complete discussion of the different species of surfaces of the third degree having double points.

1. Let the surface have one conical point. Six of the right lines pass through that point, being the lines in which the tangent cone at that point meets the surface. Fifteen of the treble tangent planes pass through the conical point, being the planes of each pair of the six lines. Each of these planes contains a right line not passing through the double point. These latter lines are such that each is intersected by six of the rest, and thus give rise to fifteen treble tangent planes not passing through the double point.

We have then $6 \times 2 + 15 = 27$; $15 \times 2 + 15 = 45$.

2. Let the surface have two conical points. The right lines are then, the line joining the points (1×4), four through each of the conical points (8×2), and seven which pass through neither of them (7×1). One of these seven is intersected by the six remaining, which lie in three different planes passing through it.

The treble tangent planes are, these three planes (3×1), six through each of the conical points (12×2), four through both conical points (4×4), and one which touches the surface along the whole length of the line joining the conical points. This latter plane only counts for two.

3. Let the surface have three conical points. The right lines then are the three joining the points (3×4), two through each of them (6×2), and three which do not meet any of them (3×1). The planes are the plane of these three lines (1), a plane touching along the whole length of each of the lines through two conical points (3×2), a plane through each of the conical points (3×2), six planes each through two (6×4), and one through the three conical points (8).

4. Let the surface have four conical points. The right lines are the six edges of the pyramid formed by these points (6×4), together with three in the same plane, each of which meets two opposite edges of the pyramid.

The planes are, the plane of these three lines (1), six planes each through one of these lines and through an edge of the pyramid (6×2), and the faces of the pyramid (4×8).

In all these cases the second method (p. 119) only gives those right lines which do not pass through double points.

5. Let the surface have a double point for which the tangent cone resolves itself into two planes. The right lines are the three in which each of these planes cuts the surface (6×3), together with nine not passing through the double point. Each of these is met by four of the rest. The planes are the two tangent planes through the double point (2×6), nine passing through that point (9×3), and six not passing through it (6×1).

6. Let the surface have one conical and one biplanar double point.

The right lines are, 3 through neither double point (3×1) no two of which intersect, three through the conical point (3×2), four through the biplanar point (4×3) and the line joining the points (1×6).

The planes are the two tangent planes at the biplanar point (2×6), three others through that point (3×3), three through the conical point (3×2), and three through both points (3×6).

7. Let the surface have two biplanar points. The right lines are three through each of these points (6×3), and the line joining them (1×9).

The planes are the tangent planes at the two points, one plane being common to both (3×6), and three planes through the two points (3×9).

8. Let the surface have one biplanar and two conical points. The lines are the sides of the triangle formed by the points (2×6), (1×4); two through the biplanar point (2×3), one through each of the conical points (2×2), and a line passing through no double point.

The planes are, the plane of the double points (12), the two tangent planes at the biplanar point (2×6), one passing through the biplanar and each of the conical points (2×6), one through both conical points (1×4), one through the biplanar alone (3), and one touching the surface along the line joins the conical points (2).

9. Let the surface have two biplanar and one conical point. The lines are the sides of the triangle formed by the points (2×6), (1×9), and one through each of the biplanar points (2×3). The planes are, the plane of the points (18), three planes which touch at biplanar points (3×6), and a plane through the two biplanar points (9).

10. Let the surface have three biplanar points. The lines are the sides of the triangle formed by the points (3×9). The planes are the plane of the points (27), and three planes touching at biplanar points (3×6).

11. Let the surface have a double point at which the tangent cone reduces itself to two coincident planes. Such a point appears to arise from the union of three conical points. There are three lines in the double plane (3×8), and three others each meeting one of the former. The planes are, the plane of these last three lines, three planes touching along the whole length of each of the former three lines $(3 \times 4\,?)$, and the double plane $(32\,?)$.

If two of the right lines in the double plane should coincide, there will be only one right line in the surface which does not meet the double point: and if the three right lines should coincide, there will be no right line which does not meet the double point.

VI.

I proceed now to investigate a general problem of which the determination of the number of right lines on a surface of the third degree is a particular case. We know that it is in general possible at every point of a surface to draw two right lines which shall meet the surface in three consecutive points; but there will be a determinate series of points through which a line can be drawn to meet the surface in four consecutive points. It is required to determine the degree of this locus, and also the degree of the surface generated by such right lines. For the particular case where the surface is of the third degree, (since a right line which meets the surface in four points must lie in it altogether), the locus will consist simply of the right lines in the surface. In this article I follow the methods used by Mr. Cayley in his memoir on Elimination and on the Theory of Curves (*Crelle*, vol. 34, p. 30).

If the coordinates of two points be x, y, z, w; x', y', z', w'; then $lx + mx'$, $ly + my'$, $lz + mz'$, $lw + mw'$, are the coordinates of a point on the line joining the two former. For any equation of the form $Ax + By + Cz + Dw$, which is satisfied

for the first two, will be satisfied for the third. Consequently if in the equation of any surface considered as a homogeneous function of four variables we substitute for x, $lx + mx'$, &c., the resulting equation solved for $\frac{l}{m}$ will give the coordinates of the points where the line joining the two given points meets the surface.

If in the condition that this equation in $\frac{l}{m}$ should have two equal roots we consider x, y, z, w, variable, we have the locus of a point such that the line joining it to the given point $x'y'z'w'$ always touches the surface; or, in other words, we have the equation of the tangent cone drawn to the surface from $(x'y'z'w')$. (Joachimsthal's Theorem, cited in the memoir referred to.)

The original equation being $U = 0$, the result of this substitution is

$$l^n U + l^{n-1} m\, \delta U + \frac{1}{1.2} l^{n-2} m^2 . \delta^2 U + \frac{1}{1.2.3} l^{n-3} m^3 \delta^3 U + \ldots .$$

$$+ m^n U' + m^{n-1} l . \delta' U' + \frac{1}{1.2} m^{n-2} l^2 . \delta^2 U + \frac{1}{1.2.3} l n^{n-3} l^3 \delta^3 U' + \ldots = 0,$$

where the symbol

$$\delta = x' \frac{d}{dx} + y' \frac{d}{dy} + z' \frac{d}{dz} + w' \frac{d}{dw},$$

$$\delta' = x \frac{d}{dx'} + y \frac{d}{dy'} + z \frac{d}{dz'} + w \frac{d}{dw'}.$$

Now in order that four of the points should coincide with the point $x'y'z'w'$, we have the conditions

$$U' = 0, \quad \delta U' = 0, \quad \delta^2 U' = 0, \quad \delta^3 U' = 0.$$

The first equation only denotes that the point should lie on the surface: in order that it should be possible to satisfy the next three, whatever be $xyzw$, we must seek the condition that the tangent plane (δu) and the next two polar surfaces $(\delta^2 u, \delta^3 u)$ should have a line in common.

Join to these three equations the equation of an arbitrary plane $\alpha x + \beta y + \gamma z + \delta w = 0$, and eliminate $xyzw$; this, which denotes the condition that the four surfaces should intersect, must be satisfied, whatever be $\alpha\beta\gamma\delta$, when the three first surfaces have a line in common. This condition is of the 6^{th} degree in $\alpha\beta\gamma\delta$, of the sixth in the coefficients of δu, of the 3^{rd} in the coefficients of $\delta^2 u$, and of the 2^{nd} in the coefficients

of $\delta^3 u$. Consequently it is of the $11n - 18$ degree in $x'y'z'w'$, and of the 11^{th} degree in the coefficients of the given surface. I say moreover that this condition is of the form

$$\Lambda u' + (\alpha x' + \beta y' + \gamma z' + \delta w')^6 \Pi = 0.$$

For in the general case let the coordinates of the six points of intersection of δu, $\delta^2 u$, $\delta^3 u$, be

$$(x_1 y_1 z_1 w_1)\,(x_2 y_2 z_2 w_2), \text{ \&c.},$$

and the condition is

$$\Pi.(\alpha x_1 + \beta y_1 + \gamma z_1 + \delta w_1)(\alpha x_2 + \beta y_2 + \gamma z_2 + \delta w_2), \text{ \&c.} = 0.$$

But when $x'y'z'w'$ satisfy the equation $u' = 0$, these six points all coincide with the point $x'y'z'w'$, and the condition must therefore reduce to

$$\Pi.(\alpha x' + \beta y' + \gamma z' + \delta w')^6 = 0.$$

The equation $\Pi = 0$ therefore which it was required to find, is of the $11n - 24$ degree in $x'y'z'w'$, and of the 11^{th} in the coefficients of the surface. The required locus is therefore a curve of the degree $11(un - 24)$. For $n = 3$, the degree $= 27$.

The equation of the surface which is the locus of right lines meeting the surface in four consecutive points, is found at once by eliminating $x'y'z'w'$ from the four equations

$$u' = 0, \quad \delta u' = 0, \quad \delta^2 u' = 0, \quad \delta^3 u' = 0,$$

and is therefore of the degree

$$n(n-2)(n-3) + 2n(n-1)(n-3) + 3n(n-1)(n-2) = 6n^3 - 22n^2 + 18.$$

But it is easy to see that the result of elimination must be of the form $u^6.M = 0$.

The equation $M = 0$ is therefore only of the degree

$$2(n-3)(3n-2).$$

In the coefficients of the surface it is of the degree

$$n(4n^2 - 18n + 22).$$

Trinity College, Dublin, October 15*th*, 1849.

ON THE SYMBOLICAL VALUE OF THE INTEGRAL $\int x^{-1}dx$.

By the Rev. W. Center.

It has often been remarked by analysts, that the singular case of the integral

$$\int x^{m-1}dx = \frac{x^m}{m} + C$$

when $m = 0$, would, at the very outset of the calculus, have occasioned much perplexity, had not the properties of the function *log* x been previously known. It thus happened that at an early stage there was at once furnished the *real* interpretation of a *symbolical* form; and this instance of discontinuity henceforth disappeared.

It is the object of this paper to recal attention to the symbolism involved in the integral

$$\int x^{-m-1}dx = \frac{x^{-m}}{-m} + C'$$

when $m = 0$, that we may obtain a more distinct conception of the definite integral

$$\int_0^\infty e^{-ax}a^{-n-1}da \text{ when } n \text{ is integer.}$$

It is obvious enough that

$$\int x^{-1}dx = -\tfrac{1}{0} + C_0' = \log x.$$

And here it will be convenient to use a proper symbol for $(\frac{1}{0})$, which is readily found in $\Gamma\overline{0}$; hence

$$\log x = -\Gamma\overline{0}\,.\,x^0 + C_0' \ldots\ldots\ldots\ldots (1).$$

Proceeding now by successive integration,

$$\int dx \log x = +\Gamma\overline{-1}\,.\,x + C_1'x + C_0,$$

$$\int^2 dx^2 \log x = -\Gamma\overline{-2}\,.\,x^2 + C_2'x^2 + C_1x + C_0,$$

$$\vdots \qquad\qquad \vdots$$

$$\int^n dx^n \log x = (-1)^{n+1}\Gamma\overline{-n}\,.\,x^n + C_n'x^n + C_{n-1}x^{n-1} + \ldots + C_1x + C_0.$$

But the terms $C_{n-1}x^{n-1} + \ldots + C_0$ belong entirely to the complementary function; which, in estimating the form of the principal function, may for the present be rejected. Hence we have the following functional expression complete in itself,

$$\left(\frac{d}{dx}\right)^{-n} \log x = (-1)^{n+1}\,\Gamma\overline{-n}\,.\,x_n + C_n'x^n \ldots\ldots\ldots (2).$$

This equation admits of easy reduction; for

$$\left(\frac{d}{dx}\right)^{-n} \log x = \{(-1)^{n+1} \Gamma(-n) + C_n'\}x^n,$$

$$= \left(\frac{-\Gamma(0) \, . \, x^0}{\Gamma(n+1)} + C_n'\right) x^n.$$

Take $A_n = C_0' - \Gamma(n+1)\, C_n'$, and substitute for C_n'; when

$$\left(\frac{d}{dx}\right)^{-n} \log x = \frac{x^n}{\Gamma(n+1)} (-\Gamma(0) \, . \, x^0 + C_0' - A_n),$$

$$= \frac{x^n}{\Gamma(n+1)} (\log x - A_n) \text{ by } (1) \ldots\ldots(3);$$

a well-known expression, in which $A_n = \left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right)$. It is thus rendered manifest, that the constants C_0', C_1',... C_n' are *discontinuous*.

The following scheme will present a distinct view of the value of $\int_0^\infty e^{-ax} \, a^{-n-1} \, da$ when n is integer.

Since $\displaystyle\int_0^\infty e^{-ax} \, da = \frac{1}{x}$, we have successively

$$\int dx \int_0^\infty e^{-ax} \, da = \log x,$$

$$= -\int_0^\infty \overset{\dagger}{e^{-ax}} \, a^{-1} \, da = -\overset{\dagger}{\Gamma(0)} \, . \, x^0 + C_0';$$

again, $\displaystyle\left(\frac{d}{dx}\right)^{-1} \int dx \int_0^\infty e^{-ax} \, da = \left(\frac{d}{dx}\right)^{-1} \log x,$

$$= +\int_0^\infty \overset{\dagger}{e^{-ax}} \, a^{-2} \, da = +\overset{\dagger}{\Gamma(-1)} \, . \, x + C_1'x;$$

$$\vdots \qquad\qquad \vdots$$

$$\left.\begin{aligned} &\text{and} \quad \left(\frac{d}{dx}\right)^{-n} \int dx \int_0^\infty e^{-ax} \, da = \left(\frac{d}{dx}\right)^{-n} \log x, \\ &= (-1)^{n+1} \int_0^\infty \overset{\dagger}{e^{-ax}} a^{-n-1} \, da = (-1)^{n+1} \overset{\dagger}{\Gamma(-n)} \, . \, x^n + C_n' \, x^n, \\ &= \frac{x^n}{\Gamma(n+1)} (\overset{\dagger}{\log} x - A_n). \end{aligned}\right\} \ldots(4).$$

Here the sign (†) marks the order of derivation; the equalities on the right-hand being the consequence of (2). Thus it appears that

$$\int_0^\infty e^{-ax} \, a^{-n-1} \, da = \Gamma(-n) \, . \, x^n + (-1)^{n+1} \, C_n'x^n \ldots\ldots(5);$$

the right-hand member of which is capable of a *real* interpretation, while in its present symbolical form it fulfils the condition of limits peculiar to the integral

$$\int_0^\infty e^{-ax}a^{n-1}da = \Gamma n . x^{-n},$$

when n is either 0 or a negative whole number.

There are some determinate properties of the constants $C'_0 \ldots C'_n$, that require examination. It is sufficiently obvious, that the order of their *relative* magnitude is indicated by

$$C'_0 > C'_1 > C'_2 \ldots > C'_{n-1} > C'_n \ldots\ldots\ldots (6).$$

We can also fix the limits of the *absolute* magnitude of C'_0. For as C'_0 made its first appearance as the constant of integration, so also we know that it must again disappear by the direct process of differentiation; or (which amounts to the same thing) its value must be such that $C'_0 \times 0 = 0$. Hence it follows from (6) that

$$C'_1 \times 0 = 0, \quad C'_2 \times 0 = 0 \ldots, \quad C'_n \times 0 = 0 \ldots (7).$$

Again, it is easy to see that $C'_0 \ldots C'_n$ are expressible in the form of definite integrals; for we have

$$A_n = \int_0^1 \frac{1-x^n}{1-x}dx = \int_0^1 \frac{dx}{1-x} - \int_0^1 \frac{x^n dx}{1-x} = C'_0 - \Gamma\overline{n+1}\, C'_n;$$

from which it follows in this case that

$$C'_0 = \int_0^1 \frac{dx}{1-x} \text{ and } C'_n = \frac{1}{\Gamma\overline{n+1}} \int_0^1 \frac{x^n dx}{1-x} \ldots (8).$$

The basis of the foregoing process rests thus. If we should attempt on purely algebraical principles to assign the form of the integral $\int x^{-1}dx$, considered as the value of $\int x^{-m-1}dx$ when m is diminished without limit, we could only infer that

$$\int \frac{dx}{x} = -\Gamma\overline{0} . x^0 + C'_0;$$

a *singular* analytical value, the real interpretation of which we know from *à priori* considerations to be $\log x$. But again we know, that the complete real integral

$$\int \frac{dx}{x} = \log x + C;$$

hence symbolically we have

$$\int \frac{dx}{x} = (-\Gamma\overline{0} . x^0 + C'_0) + C \ldots\ldots\ldots (9).$$

The interpretation of this result appears to be, that the *total* constant of integration may be divided into two parts, the

one forming an essential element of the transcendental function $\log x$, the other serving to determine the limits of the said function.

The following instance will shew the applicability of these symbolical expressions to analytical investigation. Let us take

$$\Sigma\left\{\left(\frac{d}{dx}\right)^{-n}\log x\right\} = \Sigma\,\{(-1)^{n+1}\lceil\overline{-n}\,x^n + C'_n x^n\},$$

where the symbol of summation refers to all integer values of n from 0 to ∞. On the first side

$$\Sigma\left\{\left(\frac{d}{dx}\right)^{-n}\log x\right\} = \left\{1 + \left(\frac{d}{dx}\right)^{-1} + \left(\frac{d}{dx}\right)^{-2} + \ldots\right\}\log x$$

$$= \left(\frac{d}{dx} - 1\right)^{-1}.\ \frac{d}{dx}\log x = \left(\frac{d}{dx} - 1\right)^{-1}\frac{1}{x}$$

$$= e^x\int\frac{e^{-x}}{x}\,dx.$$

On the second side we have for the first term

$$\Sigma\,\{(-1)^{n+1}\lceil\overline{-n}.x^n\} = -\lceil\overline{0}.x^0\left(1 + x + \frac{x^2}{1.2} + \ldots\right)$$

$$= -\,e^x.\lceil\overline{0}\,.\,x^0;$$

and since $C'_n = \dfrac{C'_0 - A_n}{\lceil\overline{n+1}}$, we have for the second term

$$\Sigma\,(C'_n x^n) = \Sigma\left(\frac{C'_0 - A_n}{\lceil\overline{n+1}}\,x^n\right)$$

$$= C'_0 + (C'_0 = A_1)\,x + (C'_0 - A_2)\,\frac{x^2}{1.2} + \ldots$$

$$= e^x\cdot C'_0 - \left(A_1x + \frac{A_2x^2}{1.2} + \frac{A_3x^3}{1.2.3} + \ldots\right).$$

Collecting and equating the terms,

$$e^x.\int\frac{e^{-x}}{x}\,dx = e^x.(-\lceil\overline{0}.x^0 + C'_0) - \left(A_1x + \frac{A_2x^2}{1.2} + \ldots\right);$$

or
$$\int\frac{e^{-x}}{x}\,dx = \log x - e^{-x}.\left(A_1x + \frac{A_2x^2}{1.2} + \frac{A_3x^3}{1.2.3} + \ldots\right)$$

$$= \log x - x + \frac{x^2}{1.2^2} - \frac{x^3}{1.2.3^2} + \frac{x^4}{1.2.3.4^2} - \&c.$$

This is a well-known case of common integration.

Longside, Mintlaw, Oct. 17*th*, 1849.

SINGULAR APPLICATION OF GEOMETRY OF THREE DIMENSIONS TO A PLANE PROBLEM.

By G. W. Hearn, of the Royal Military College.

Let u, v, w be three linear functions of x, y, each containing three arbitrary constants. Three other linear functions, r, s, t, of x, y, may be found, in an infinite number of ways which shall satisfy the identities

$$r + s + t = u + v + w \ldots\ldots\ldots\ldots (1),$$

and

$$r^2 + s^2 + t^2 = u^2 + v^2 + w^2 \ldots\ldots\ldots\ldots (2).$$

Suppose r, s, t, expressed in terms of u, v, w, are

$$\left.\begin{aligned} r &= a_1u + b_1v + c_1w \\ s &= a_2u + b_2v + c_2w \\ t &= a_3u + b_3v + c_3w \end{aligned}\right\} \ldots\ldots\ldots\ldots (3).$$

The identities (1) and (2) require the following nine equations:

$$\left.\begin{aligned} a_1 + a_2 + a_3 &= 1 \\ b_1 + b_2 + b_3 &= 1 \\ c_1 + c_2 + c_3 &= 1 \end{aligned}\right\} \ldots\ldots\ldots\ldots (4),$$

$$\left.\begin{aligned} a_1^2 + a_2^2 + a_3^2 &= 1 \qquad & a_1b_1 + a_2b_2 + a_3b_3 &= 0 \\ b_1^2 + b_2^2 + b_3^2 &= 1 \qquad & a_1c_1 + a_2c_2 + a_3c_3 &= 0 \\ c_1^2 + c_2^2 + c_3^2 &= 1 \qquad & b_1c_1 + b_2c_2 + b_3c_3 &= 0 \end{aligned}\right\} \ldots (5).$$

Now though the preceding nine equations contain nine quantities, yet these are not absolutely determined; but if any one of them is assumed, the rest are known. This I proceed to shew.

Let there be a sphere, cone and plane, whose equations referred to rectangular axes are

$$x^2 + y^2 + z^2 = 1,$$

$$xy + xz + yz = 0,$$

and

$$x + y + z = 1.$$

Any of these equations being a consequence of the other two, it follows that the sphere and cone intersect in a circle which lies in the plane. The cone is one of revolution having for its vertex the centre of the sphere, and for its axis the line whose equations are $x = y = z$. Each of the coordinate axes is a generating line of the cone; and by imagining them to revolve round the axis of the cone so as to continue at right angles to each other, and also to preserve the same inclination to the axis of the cone, each of these

moveable axes will generate the conical surface. Now, considering any position of the system of moveable axes, let $(a'a''a''')$, $(b'b''b''')$, and $(c'c''c''')$ denote the points in which these axes respectively intersect the plane or sphere, therefore

$$\begin{aligned} a' + a'' + a''' &= 1, \\ b' + b'' + b''' &= 1, \\ c' + c'' + c''' &= 1, \\ a'^2 + a''^2 + a'''^2 &= 1, \\ b'^2 + b''^2 + b'''^2 &= 1, \\ \text{and } c'^2 + c''^2 + c'''^2 &= 1. \end{aligned}$$

Also since the moveable axes are mutually rectangular, we have

$$\begin{aligned} a'b' + a''b'' + a'''b''' &= 0, \\ a'c' + a''c'' + a'''c''' &= 0, \\ b'c' + b''c'' + b'''c''' &= 0. \end{aligned}$$

Now by comparing the last nine equations with (4) and (5), it is evident that the quantities $a'a''a'''$ $b'b''b'''$ $c'c''c'''$ may be taken to denote $a_1a_2a_3$ $b_1b_2b_3$ $c_1c_2c_3$ respectively. But the former nine quantities admit of an infinite number of systems of values (corresponding to the different positions of the moveable axes), and hence the same is true of the latter system of quantities. It is clear, however, that if any value be assigned to any one of these quantities, the other eight will be determined.

From (3) and (5), we have

$$\begin{aligned} u &= a_1r + a_2s + a_3t, \\ v &= b_1r + b_2s + b_3t, \\ w &= c_1r + c_2s + c_3t, \end{aligned}$$

and hence, (4), the point determined by $r = s = t$, is the same as that determined by $u = v = w$.

Let us apply these conclusions.

We know that

$$uv + uw + vw = 0 \ldots\ldots\ldots\ldots (6),$$

and $$u^2 + v^2 + w^2 - 2uv - 2uw - 2vw = 0 \ldots\ldots (7),$$

represent conic sections respectively circumscribed about and inscribed in the triangle uvw. Now from the identities (1) and (2), we have also

$$rs + rt + st = uv + uw + vw$$

identically; and hence the equations,

$$rs + rt + st = 0 \ldots\ldots\ldots\ldots\ldots (8),$$

and

$$r^2 + s^2 + t^2 - 2rs - 2rt - 2st = 0 \ldots\ldots\ldots (9),$$

are respectively identical with (6) and (7). Hence we have the same conic section, (6) or (8), circumscribed about both the triangles uvw, rst, and the same conic section, (7) or (9), inscribed in both triangles.

Moreover, the lines joining the angular points of the triangle uvw with the points of contact of the opposite sides, are $u = v$, $v = w$, and $w = u$ respectively, which intersect in the point $u = v = w$; also $r = s$, $s = t$, and $t = r$ are the lines joining the angular points of the triangle rst with the points of contact of the opposite sides, and these lines pass through $r = s = t$, which is the same point as $u = v = w$. Again, $u + v = 0$ is the equation of the tangent to the circumscribed conic touching at the angular point $u = v = 0$; similarly, $v + w = 0$, and $w + u = 0$, are the equations to the tangents at the points $v = w = 0$, and $w = u = 0$, respectively; hence the tangents at the angular points of the triangle uvw intersect the opposite sides in three points in the straight line $u + v + w = 0$. In like manner, the tangents drawn at the angular points of the triangle rst intersect the opposite sides in three points in the straight line $r + s + t = 0$, which is the same line as $u + v + w = 0$. Now uvw is a fixed triangle, and rst is a moveable one; hence,

If any conic section be inscribed in a given triangle, a conic section may be circumscribed about the same, so as to fulfil the following three conditions:

1. *An infinite number of triangles may be drawn so as to be circumscribed about the first conic and inscribed in the second;*

2. *The lines drawn from the angular points of each of these triangles to the points of contact of the opposite sides shall always pass through the same fixed point:*

3. *The tangents to the circumscribed conic drawn at the angular points of each triangle shall intersect the opposite sides in three points in the same fixed straight line.*

The fixed point may with propriety be called the common pole of the system, and the fixed line the common axis.

It is to be observed that this is not the first time that the geometrical relations of the six conditions (5) have been brought into operation for the purpose of avoiding troublesome eliminations. This was done by Mr. Weddle, in a paper entitled "Investigation of certain Properties of the Ellipsoid," published in this *Journal*, vol. II. New Series, p. 13.

NOTE ON THE MOTION OF ROTATION OF A SOLID OF REVOLUTION.

By Arthur Cayley.

Using the notation employed in my former papers on the subject of rotation (*Journal*, vol. III. p. 224; and New Series, vol. I. pp. 167, 264), suppose $B = A$, then r is constant, equal to n suppose; and writing

$$\frac{(A - C)\,n}{A} = \nu, \quad \text{or} \quad C = \left(1 - \frac{\nu}{n}\right) A.$$

Also putting $$\theta = \nu t + \gamma,$$

(where γ is an arbitrary constant) the values of p, q, r are easily seen to be given by the equations

$$\begin{cases} p = M \sin\theta, \\ q = M \cos\theta, \\ r = n, \end{cases}$$

(where M is arbitrary). And consequently

$$h = A\{M^2 + n(n - \nu)\},$$
$$k^2 = A^2\{M^2 + (n - \nu)^2\}.$$

Also, since $\mathrm{a}^2 + \mathrm{b}^2 + \mathrm{c}^2 = k^2$, we may write

$$\mathrm{a} = -\,k \sin i \cos j,$$
$$\mathrm{b} = \quad k \cos i \cos j,$$
$$\mathrm{c} = \quad k \sin j;$$

k having the value above given, and the angles i, j being arbitrary.

From the equations (12) and (15) in the second of the papers quoted, we deduce

$$2v = k\{k + A(n - \nu)\sin j + MA \cos j \cos(\theta + i)\},$$
$$\Phi = k\{n \sin j + M \cos j \cos(\theta + i)\},$$
$$\nabla = -\,\nu k M A \cos j \sin(\theta + i),$$

(values which verify as they should do the equation (19)). Hence, from the equation (27), writing $\dfrac{2dv}{\nabla} = dt = \dfrac{1}{\nu}\,d\theta$, we have

$$2\tan^{-1}\frac{\Omega}{k} = \delta + \frac{1}{\nu}\int d\theta\,\frac{h + kn \sin j + kM \cos j \cos(\theta + i)}{k + A(n - \nu)\sin j + MA \cos j \cos(\theta + i)}.$$

This is easily integrated; but the only case which appears likely to give a simple result is when the quantity under the

integral sign is constant, or

$$A(h + kn \sin j) = k\{k + A(n - \nu)\sin j\},$$

or

$$Ah - k^2 + Ak\nu \sin j = 0;$$

that is,

$$A(n - \nu) + k \sin j = 0:$$

whence

$$\sin j = -\frac{A(n-\nu)}{k},\ \cos j = \frac{AM}{k},\ \text{or}\ \tan j = \frac{-(n-\nu)}{M} = -\frac{C}{A}\cdot\frac{n}{M}.$$

Or observing that $\frac{\pi}{2} - j$ is the inclination of the axis of z to the normal to the invariable plane, this equation shews that the supposition above is not any restriction upon the generality of the motion, but amounts only to supposing that the axis of z (which is a line fixed in space) is taken upon the surface of a certain right cone having the perpendicular to the invariable plane for its axis. Resuming the solution of the problem, we have

$$2\tan^{-1}\frac{\Omega}{k} = \delta + \frac{k}{\nu A}\,\theta,$$

which may also be written under the form

$$2\tan^{-1}\frac{\Omega}{k} = \delta_1 + \frac{kt}{A},$$

$\left(\text{where } \delta_1 = \delta + \frac{k\gamma}{\nu A}\right)$. And hence

$$\Omega = k\tan\tfrac{1}{2}\left(\delta_1 + \frac{kt}{A}\right) = k\tan\psi,$$

where

$$\psi = \tfrac{1}{2}\left(\delta_1 + \frac{kt}{A}\right).$$

Substituting these values,

$$\mathrm{a} = -MA\sin i,\quad \mathrm{b} = MA\cos i,\quad \mathrm{c} = -A(n-\nu),$$

$$2v = A^2M^2\{1 + \cos(\theta + i)\} = 2M^2A^2\cos^2\tfrac{1}{2}(\theta + i).$$

And substituting in the equations (14) the values of λ, μ, ν reduce themselves to

$$\begin{cases}\lambda = \dfrac{1}{MA\cos\frac{1}{2}(\theta+i)}\{k\tan\psi\sin\tfrac{1}{2}(\theta-i) - A(n-\nu)\cos\tfrac{1}{2}(\theta-i)\},\\[2ex] \mu = \dfrac{1}{MA\cos\frac{1}{2}(\theta+i)}\{k\tan\psi\cos\tfrac{1}{2}(\theta-i) + A(n-\nu)\sin\tfrac{1}{2}(\theta-i)\},\\[2ex] \nu = \tan\tfrac{1}{2}(\theta+i);\end{cases}$$

where, recapitulating, $\theta = \nu t + \gamma$, $2\psi = \frac{kt}{A} + \delta_1$.

I may notice, in connexion with the problem of rotation, a memoir, "Specimen Inaugurale de motu gyratorio corporis rigidi, &c.," by A. S. Rueb (Utrecht, 1834), which contains some very interesting developments of the ordinary solution of the problem, by means of the theory of elliptic functions.

ON A SYSTEM OF EQUATIONS CONNECTED WITH MALFATTI'S PROBLEM, AND ON ANOTHER ALGEBRAICAL SYSTEM.

By Arthur Cayley.

Consider the equations

$$by^2 + cz^2 + 2fyz = \theta^2 a(bc - f^2),$$
$$cz^2 + ax^2 + 2gzx = \theta^2 b(ca - g^2),$$
$$ax^2 + by^2 + 2hxy = \theta^2 c(ab - h^2).$$

Or, as they may be more conveniently written,

$$by^2 + cz^2 + 2fyz = \theta^2 a\mathfrak{A},$$
$$cz^2 + ax^2 + 2gzx = \theta^2 b\mathfrak{B},$$
$$ax^2 + by^2 + 2hxy = \theta^2 c\mathfrak{C}.$$

The second and third equations give

$$(g^2\mathfrak{C} - h^2\mathfrak{B})x^2 - b^2\mathfrak{B}y^2 + c^2\mathfrak{C}z^2 + 2cg\mathfrak{C}zx - 2bh\mathfrak{B}xy = 0,$$

or $\{(g^2\mathfrak{C} - h^2\mathfrak{B})x - bh\mathfrak{B}y + cg\mathfrak{C}z\}^2 - \mathfrak{B}\mathfrak{C}(-bgy + chz)^2 = 0;$

i.e. $(g^2\mathfrak{C} - h^2\mathfrak{B})x - b\sqrt{\mathfrak{B}}(g\sqrt{\mathfrak{C}} + h\sqrt{\mathfrak{B}})y + c\sqrt{\mathfrak{C}}(g\sqrt{\mathfrak{C}} + h\sqrt{\mathfrak{B}})z = 0.$

Or dividing by $g\sqrt{\mathfrak{C}} + h\sqrt{\mathfrak{B}}$, and writing down the system of equations to which the equation thus obtained belongs,

$$(g\sqrt{\mathfrak{C}} - h\sqrt{\mathfrak{B}})\,x - b\sqrt{\mathfrak{B}}y + c\sqrt{\mathfrak{C}}z = 0,$$
$$a\sqrt{\mathfrak{A}}x + (h\sqrt{\mathfrak{A}} - f\sqrt{\mathfrak{C}})\,y - c\sqrt{\mathfrak{C}}z = 0,$$
$$-a\sqrt{\mathfrak{A}}x + b\sqrt{\mathfrak{B}}y + (f\sqrt{\mathfrak{B}} - g\sqrt{\mathfrak{A}})\,z = 0.$$

Whence also

$$(h\sqrt{\mathfrak{A}} + b\sqrt{\mathfrak{B}} - f\sqrt{\mathfrak{C}})\,y - (g\sqrt{\mathfrak{A}} - f\sqrt{\mathfrak{B}} + c\sqrt{\mathfrak{C}})\,z = 0,$$
$$(-g\sqrt{\mathfrak{A}} + f\sqrt{\mathfrak{B}} + c\sqrt{\mathfrak{C}})\,z - (a\sqrt{\mathfrak{A}} + h\sqrt{\mathfrak{B}} - g\sqrt{\mathfrak{C}})\,x = 0,$$
$$(a\sqrt{\mathfrak{A}} - h\sqrt{\mathfrak{B}} + g\sqrt{\mathfrak{C}})\,x - (-h\sqrt{\mathfrak{A}} + b\sqrt{\mathfrak{B}} + f\sqrt{\mathfrak{C}})\,y = 0.$$

These equations may be written

$$\frac{1}{K}\{\mathfrak{F}\surd\mathfrak{A} - \mathfrak{G}\surd\mathfrak{B} - \mathfrak{H}\surd\mathfrak{C} + \surd(\mathfrak{ABC})\}\,[\{\mathfrak{G} + \surd(\mathfrak{CA})\}\,y - \{\mathfrak{H} + \surd(\mathfrak{AB})\}\,z] = 0,$$

$$\frac{1}{K}\{-\mathfrak{F}\surd\mathfrak{A} + \mathfrak{G}\surd\mathfrak{B} - \mathfrak{H}\surd\mathfrak{C} + \surd(\mathfrak{ABC})\}\,[\{\mathfrak{H} + \surd(\mathfrak{AB})\}\,z - \{\mathfrak{F} + \surd(\mathfrak{BC})\}\,x] = 0,$$

$$\frac{1}{K}\{-\mathfrak{F}\surd\mathfrak{A} - \mathfrak{G}\surd\mathfrak{B} + \mathfrak{H}\surd\mathfrak{C} + \surd(\mathfrak{ABC})\}\,[\{\mathfrak{F} + \surd(\mathfrak{BC})\}\,x - \{\mathfrak{G} + \surd(\mathfrak{CA})\}\,y] = 0\,;$$

where, as usual,

$$\mathfrak{F} = gh - af,\quad \mathfrak{G} = hf - bg,\quad \mathfrak{H} = fg - ch,$$

$$K = abc - af^2 - bg^2 - ch^2 + 2fgh.$$

(In fact the coefficient of y in the first equation is

$$\frac{1}{K}\{(\mathfrak{FG} - \mathfrak{CH})\surd\mathfrak{A} + (\mathfrak{AC} - \mathfrak{G}^2)\surd\mathfrak{B} - (\mathfrak{GH} - \mathfrak{AF})\surd\mathfrak{C}\} = h\surd\mathfrak{A} + b\surd\mathfrak{B} - f\surd\mathfrak{C},$$

as it should be, and similarly for the coefficients of the remaining terms). We have therefore

$$\{\mathfrak{F} + \surd(\mathfrak{BC})\}\,x = \{\mathfrak{G} + \surd(\mathfrak{CA})\}\,y = \{\mathfrak{H} + \surd(\mathfrak{AB})\}\,z\,;$$

or, what comes to the same thing,

$$yz = \tfrac{1}{2}s\,\{\mathfrak{F} + \surd(\mathfrak{BC})\},$$

$$zx = \tfrac{1}{2}s\,\{\mathfrak{G} + \surd(\mathfrak{CA})\},$$

$$xy = \tfrac{1}{2}s\,\{\mathfrak{H} + \surd(\mathfrak{AB})\}.$$

Now $a\,\{\mathfrak{G} + \surd(\mathfrak{AC})\}\,\{\mathfrak{H} + \surd(\mathfrak{AB})\}$

$$= \{\mathfrak{F} + \surd(\mathfrak{BC})\}\,\{abc - fgh + f\surd(\mathfrak{BC}) - g\surd(\mathfrak{CA}) - h\surd(\mathfrak{AB})\},$$

$b\,\{\mathfrak{H} + \surd(\mathfrak{AB})\}\,\{\mathfrak{F} + \surd(\mathfrak{BC})\}$

$$= \{\mathfrak{G} + \surd(\mathfrak{CA})\}\,\{abc - fgh - f\surd(\mathfrak{BC}) + g\surd(\mathfrak{CA}) - h\surd(\mathfrak{AB})\},$$

$c\,\{\mathfrak{F} + \surd(\mathfrak{BC})\}\,\{\mathfrak{G} + \surd(\mathfrak{CA})\}$

$$= \{\mathfrak{H} + \surd(\mathfrak{AB})\}\,\{abc - fgh - f\surd(\mathfrak{BC}) - g\surd(\mathfrak{CA}) + h\surd(\mathfrak{AB})\},$$

{as readily appears by writing the first of these equations under the form

$a\,\{\mathfrak{G} + \surd(\mathfrak{AC})\}\,\{\mathfrak{H} + \surd(\mathfrak{AB})\}$

$$= \{\mathfrak{F} + \surd(\mathfrak{BC})\}\,\{a\mathfrak{A} - f\mathfrak{F} + f\surd(\mathfrak{BC}) - g\surd(\mathfrak{CA}) - h\surd(\mathfrak{AB})\}$$

and comparing the rational term and the coefficients of $\surd(\mathfrak{BC})$, $\surd(\mathfrak{CA})$, $\surd(\mathfrak{AB})$}.

Hence, observing the values of yz, zx, xy,

$$x^2 = \frac{s}{2a}\left\{abc - fgh + f\sqrt{(\mathfrak{B}\mathfrak{C})} - g\sqrt{(\mathfrak{C}\mathfrak{A})} - h\sqrt{(\mathfrak{A}\mathfrak{B})}\right\},$$

$$y^2 = \frac{s}{2b}\left\{abc - fgh - f\sqrt{(\mathfrak{B}\mathfrak{C})} + g\sqrt{(\mathfrak{C}\mathfrak{A})} - h\sqrt{(\mathfrak{A}\mathfrak{B})}\right\},$$

$$z^2 = \frac{s}{2c}\left\{abc - fgh - f\sqrt{(\mathfrak{B}\mathfrak{C})} - g\sqrt{(\mathfrak{C}\mathfrak{A})} - h\sqrt{(\mathfrak{A}\mathfrak{B})}\right\}.$$

Hence, forming the value of any one of the functions

$$by^2 + cz^2 + 2fyz,\quad cz^2 + ax^2 + 2gxy,\quad ax^2 + by^2 + 2hxy,$$

we obtain $s = \theta^2$; or we have

$$\left\{\begin{aligned}
x^2 &= \frac{\theta^2}{2a}\left\{abc - fgh + f\sqrt{(\mathfrak{B}\mathfrak{C})} - g\sqrt{(\mathfrak{C}\mathfrak{A})} - h\sqrt{(\mathfrak{A}\mathfrak{B})}\right\},\\
y^2 &= \frac{\theta^2}{2b}\left\{abc - fgh - f\sqrt{(\mathfrak{B}\mathfrak{C})} + g\sqrt{(\mathfrak{C}\mathfrak{A})} - h\sqrt{(\mathfrak{A}\mathfrak{B})}\right\},\\
z^2 &= \frac{\theta^2}{2c}\left\{abc - fgh - f\sqrt{(\mathfrak{B}\mathfrak{C})} - g\sqrt{(\mathfrak{C}\mathfrak{A})} + h\sqrt{(\mathfrak{A}\mathfrak{B})}\right\},\\
yz &= \tfrac{1}{2}\theta^2\left\{\mathfrak{F} + \sqrt{(\mathfrak{B}\mathfrak{C})}\right\},\\
zx &= \tfrac{1}{2}\theta^2\left\{\mathfrak{G} + \sqrt{(\mathfrak{C}\mathfrak{A})}\right\},\\
xy &= \tfrac{1}{2}\theta^2\left\{\mathfrak{H} + \sqrt{(\mathfrak{A}\mathfrak{B})}\right\}.
\end{aligned}\right.$$

It may be remarked that the equations

$$\begin{aligned}
by^2 + cz^2 + 2fyz &= L,\\
c'z^2 + a'x^2 + 2g'yz &= M,\\
a''x^2 + b''y^2 + 2h''xy &= N,
\end{aligned}$$

in which the coefficients are supposed to be such that the functions

$$\begin{aligned}
&M(a''x^2 + b''y^2 + 2h''xy) - N(c'z^2 + a'x^2 + 2g'yz),\\
&N(by^2 + cz^2 + 2fyz) - L(a''x^2 + b''y^2 + 2h''yz),\\
&L(c'z^2 + a'x^2 + 2g'yz) - M(by^2 + cz^2 + 2fyz),
\end{aligned}$$

are each of them decomposable into linear factors, may always be reduced to a system of equations similar to those which have just been solved.

Suppose

$$f = g = h = \frac{1}{\theta^2} = \sqrt{\left(\frac{abc}{a+b+c}\right)} = r,$$

and write $\sqrt{X}, \sqrt{Y}, \sqrt{Z}$ instead of x, y, z. The equations to be solved become

$$bY + cZ + 2r\sqrt{(YZ)} = (b + c)\,r,$$
$$cZ + aX + 2r\sqrt{(ZX)} = (c + a)\,r,$$
$$aX + bY + 2r\sqrt{(XY)} = (a + b)\,r.$$

where $r^2 = \dfrac{abc}{a+b+c}$, and the solution is

$$\mathrm{x} = \frac{r}{2a}\left\{a + b + c - r + \sqrt{(r^2 + a^2)} - \sqrt{(r^2 + b^2)} - \sqrt{(r^2 + c^2)}\right\},$$

$$\mathrm{y} = \frac{r}{2b}\left\{a + b + c - r + \sqrt{(r^2 + a^2)} + \sqrt{(r^2 + b^2)} - \sqrt{(r^2 + c^2)}\right\},$$

$$\mathrm{z} = \frac{r}{2c}\left\{a + b + c - r - \sqrt{(r^2 + a^2)} - \sqrt{(r^2 + b^2)} + \sqrt{(r^2 + c^2)}\right\}.$$

$$\sqrt{(\mathrm{yz})} = \tfrac{1}{2}\left\{r - a + \sqrt{(r^2 + a^2)}\right\},$$
$$\sqrt{(\mathrm{zx})} = \tfrac{1}{2}\left\{r - b + \sqrt{(r^2 + b^2)}\right\},$$
$$\sqrt{(\mathrm{xy})} = \tfrac{1}{2}\left\{r - c + \sqrt{(r^2 + c^2)}\right\},$$

a system of formulæ which contain the solution of the problem "In a given triangle to inscribe three circles such that each circle touches the remaining two circles and also two sides of the triangle." In fact, if r denote the radius of the inscribed circle, and a, b, c the distances of the angles of the triangle from the points where the sides are touched by the inscribed circle (quantities which it is well known satisfy the condition $r^2 = \dfrac{abc}{a+b+c}$), also if x, y, z denote the radii of the required circles, there is no difficulty whatever in obtaining for the determination of x, y, z, the above system of equations. The problem in question was first proposed and solved by an Italian geometer named Malfatti, and has been called after him Malfatti's problem. His solution, dated 1803, and published in the 10th volume of the Transactions of the Italian Academy of Sciences, appears to have consisted in shewing that the values first found for the radii of the three circles satisfy the equations given above, without any indication of the process of obtaining the expressions for these radii. Further information as to the history of the problem may be found in the memoir "Das Malfattische Problem neu gelösst von *C. Adams*," Winterthur, 1846.

In connexion with the preceding investigations may be considered the problem of determining l and m from the equations

$$B(l+\theta)^2 - 2H(l+\theta)m + (A+1)m^2 = 0,$$
$$A(m+\theta)^2 - 2H(m+\theta)l + (B+1)l^2 = 0;$$

which express that the function

$$\theta^2 U + (lx+my)^2, \quad (U = Ax^2 + 2Hxy + By^2),$$

has for one of its factors a factor of $U + x^2$, and for the other of its factors a factor of $U + y^2$. There is no difficulty in solving these equations; and if we write

$$K = AB - H^2, \quad \varpi_1 = \sqrt{(-K-B)}, \quad \varpi_2 = \sqrt{(-K-A)},$$

the result is easily shewn to be

$$l:m:\theta = A(B+H+\varpi_1) : B(A+H+\varpi_2) : (H+\varpi_1)(H+\varpi_2) - AB.$$

But the problem may be considered as the problem for two variables, analogous to that of determining the conic having a double contact with a given conic, and touching three conics each of them having a double contact with the given conic; and in this point of view I was led to the following solution. If we assume

$$Bl - Hm = u, \quad -Hl + Am = v,$$

or, what is the same thing,

$$Kl = Au + Hv, \quad Km = Hu + Bv,$$

then putting $\quad \nabla = Au^2 + 2Huv + Bv^2,$

the two equations become after some reduction

$$(u - K\theta)^2 = -\varpi_1^2\left(K\theta^2 + \frac{1}{K}\nabla\right),$$

$$(v - K\theta)^2 = -\varpi_2^2\left(K\theta^2 + \frac{1}{K}\nabla\right).$$

Hence, writing $K\theta^2 + \frac{1}{K}\nabla = -s^2$, we have

$$u = K\theta + \varpi_1 s,$$
$$v = K\theta + \varpi_2 s,$$
$$\nabla + K^2\theta^2 + Ks^2 = 0;$$

and substituting these values of u, v in the last equation,

$$A(K\theta + \varpi_1 s)^2 + 2H(K\theta + \varpi_1 s)(K\theta + \varpi_2 s) + B(K\theta + \varpi_2 s)^2$$
$$+ K^2\theta^2 + Ks^2 = 0.$$

or reducing,

$$K^2\theta^2(A+2H+B+1)+2K\theta s\left\{(A+H)\varpi_1+(H+B)\varpi_2\right\}$$
$$+s^2(A\varpi_1^2+2H\varpi_1\varpi_2+B\varpi_2^2+K)=0.$$

And thence

$$[K\theta(A+2H+B+1)+s\left\{(A+H)\varpi_1+(H+B)\varpi_2\right\}]^2$$
$$=s^2[\left\{(A+H)\varpi_1+(H+B)\varpi_2\right\}^2-(A+2H+B+1)(A\varpi_1^2+2H\varpi_1\varpi_2+B\varpi_2^2+K)]$$
$$=s^2\left\{-K(\varpi_1-\varpi_2)^2-(A\varpi_1^2+2H\varpi_1\varpi_2+B\varpi_2^2)-K(A+2H+B+1)\right\}$$
$$=s^2\left\{-(A+K)\varpi_1^2-(B+K)\varpi_2^2-K(A+2H+B+1)+2(K-H)\varpi_1\varpi_2\right\}$$
$$=s^2\left\{2\varpi_1^2\varpi_2^2-K(A+2H+B+1)+2(K-H)\varpi_1\varpi_2\right\}$$
$$=s^2(\varpi_1\varpi_2+K-H)^2:$$

and therefore

$$K\theta(A+2H+B+1)+s\left\{(A+H)\varpi_1+(H+B)\varpi_2\right\}=s(\varpi_1\varpi_2+K-H),$$

$$s=\frac{K\theta(A+2H+B+1)}{\varpi_1\varpi_2-(A+H)\varpi_1-(H+B)\varpi_2+K-H}.$$

But $\quad Kl=Au+Hv=(A+H)K\theta+(A\varpi_1+H\varpi_2)s,$

$$Km=(H+B)K\theta+(H\varpi_1+B\varpi_2)s,$$

and substituting the above value of s, we obtain, after some simple reductions,

$$l:m:\theta=(\varpi_1\varpi_2+K-H)(A+H-\varpi_2):(\varpi_1\varpi_2+K-H)(B+H-\varpi_2)$$
$$:\left\{\varpi_1\varpi_2-(A+H)\varpi_1-(B+H)\varpi_2+K-H\right\},$$

a result which presents itself in a very different form from the one previously obtained. If, however, the terms of this proportion be multiplied by the factor

$$\frac{H\varpi_1\varpi_2-K\varpi_1-K\varpi_2+AB-KH}{(A+2H+B+1)K},$$

they become (as they ought to do) identical with those of the former proportion, and the identical equations to which this process gives rise are not without interest.

ON THE MATHEMATICAL THEORY OF ELECTRICITY IN EQUILIBRIUM.

V.—EFFECTS OF ELECTRICAL INFLUENCE ON INTERNAL SPHERICAL, AND ON PLANE CONDUCTING SURFACES.

By WILLIAM THOMSON.

1. In the preceding articles of this series certain problems with reference to conductors bounded externally by spherical surfaces have been considered. It is now proposed to exhibit the solutions of similar problems with reference to the distribution of electricity on concave spherical surfaces, and on planes.

The object of the following short digression is to define and explain the precise signification of certain technical terms and expressions which will be used in this and in subsequent papers on the Mathematical Theory of Electricity.

External and Internal Conducting Surfaces.

2. DEF. 1. A closed surface separating conducting matter within it from air* without it, is called an *external conducting surface*.

DEF. 2. A closed surface separating air within it from conducting matter without it is called *an internal conducting surface*.

Thus, according to these definitions, a solid conductor has only one "conducting surface," and that "an external conducting surface."

A conductor containing within it one or more hollow spaces filled with air, possesses two or more "conducting surfaces;" namely, one "external conducting surface," and one or more "internal conducting surfaces."

A complex arrangement, consisting of a hollow conductor and other conductors insulated within it, presents several external and internal conducting surfaces; namely, an "external conducting surface" for each individual conductor, and as many "internal conducting surfaces" as there are hollow spaces in the different conductors.

3. In any arrangement such as this, there are different masses of air which are completely separated from one another by conducting matter. Now among the General Theorems alluded to in II. § 13, it will be proved that the

* See II. § 10, excluding all nonconductors except air, or gases.

bounding surface or surfaces of any such mass of air cannot experience any electrical influence from the surfaces of the other masses of air, or from any electrified bodies within them. Hence any statical phenomena of electricity which may be produced in a hollow space surrounded continuously by conducting matter,—whether this conducting envelope be a sheet even as thin as gold leaf, or a massive conductor of any external form and dimensions,—will depend solely on the form of the internal conducting surface.

4. Prop. *An internal conducting surface cannot receive a charge of electricity independently of the influence of electrified bodies within it.*

5. The demonstration of this proposition depends on what precedes, and on one of the General Theorems, already alluded to, by which it appears that it is impossible to distribute a charge of electricity on a closed surface in such a manner that there may be no resultant force exerted on external points, and consequently impossible, with merely a distribution of electricity on an internal conducting surface, to satisfy the condition of electrical equilibrium with reference to the conducting matter which surrounds it.

The preceding proposition (§ 4) is fully confirmed by experiment (Faraday's *Experimental Researches*, §§ 1173, 1174). In fact, the certainty with which its truth has been practically demonstrated in a vast variety of cases, by all electrical experimenters, may be regarded as a very strong part of the evidence on which the Elementary Laws as stated above (II. § 12) rest.

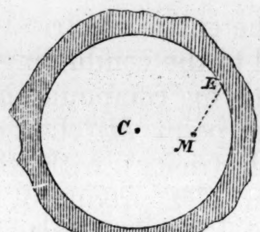
6. It might be farther stated that the total quantity of electricity produced by influence on an internal conducting surface is necessarily equal in every case to the total quantity of electricity on the influencing electrified bodies insulated within it. This will also be demonstrated among the General Theorems; but its truth in the special case which we are now to consider, will, as we shall see, be established by a special demonstration.

Electrical Influence on an Internal Spherical Conducting Surface.

7. In investigating the effects of electrical influence upon an external, or convex, spherical conducting surface (IV. §§ 5, 6, 7), we have considered the conductor to be insulated and initially charged with a given amount of electricity. In the present investigation no such considerations are necessary, since, according to the statements in the preceding

paragraphs, it is of no consequence, in the case now contemplated, whether the conductor containing the internal conducting surface be insulated or not; and it is impossible to charge this internal surface initially, or to charge it at all, independently of the influence of electrified bodies within it. With the modifications and omissions necessary on this account, the preceding investigations are applicable to the case now to be considered.

8. PROB. To find the electrical density at any point of an internal spherical conducting surface with an electrical point insulated within it.



Let m denote the quantity of electricity in the electrical point M; f its distance from C the centre of the sphere, and a the radius of the sphere.

If the expression for the electrical density at any point E of the internal surface be

$$\rho = \frac{\lambda}{ME^3},^*$$

(λ a constant); the force exerted by the electrified spherical surface on any point without it will (IV. § 2) be the same as if a quantity of matter equal to $\frac{\lambda . 4\pi a}{a^2 - f^2}$ were collected at the point M. Hence if we take λ such that

$$\frac{\lambda . 4\pi a}{a^2 - f^2} = -m,$$

the total resultant force, due to the given electrical point and to the electrified surface, will vanish at every point external to the spherical surface, and consequently at every point within the substance of the conductor; so that the condition of electrical equilibrium (II. § 12), in the prescribed circumstances, is satisfied. We conclude therefore that the required density, at any point E, of the internal spherical surface is given by the equation

$$\rho = -\frac{(a^2 - f^2)\,m}{4\pi a} \cdot \frac{1}{ME^3} \, \ldots\ldots\ldots (A).$$

* We cannot here, as in (A) of II. § 5, annex a constant term, since in this case there would result a force due to a corresponding quantity of electricity, concentrated at the centre of the sphere, on all points of the conducting mass.

This solution of the problem is complete, since it satisfies all the conditions that can possibly be prescribed, and it is unique, as follows from the general Theorem referred to in § 5.*

Cor. The total quantity of electricity produced by the influence of an electrical point within an internal spherical conducting surface is equal, but of the opposite kind, to that of the influencing point.

This follows at once from the investigation of § 4 in No. iv.; from which we also deduce the conclusion stated below in the next §.

9. The entire electrical force, which vanishes for all points external to the conducting surface, may, for points within it, be found by compounding the force due to the given influencing point M (charged, by hypothesis, with a quantity m of electricity) with that due to an imaginary point I, taken in CM produced, at such a distance from C that $CM.CI = a^2$, and charged with a quantity of electricity equal to $-\frac{a}{f}m$.

Cor. The resultant force at an internal point infinitely near the surface, is in the direction of the normal, and is equal to $4\pi\rho$, if ρ be the electrical density of the surface in the neighbourhood.

10. The mutual attraction between the influencing point M, and the surface inductively electrified, will be found as in iv. § 7, provided the *uniform supplementary distribution* which was there introduced, be omitted. Hence, omitting the term of (B) which depends on this supplementary distribution; or simply, without reference to (B), considering

* For if there were two distinct solutions there would be two different distributions on the spherical surface, each balancing on external points the action of the internal influencing body, and therefore each producing the same force at external points. Hence a distribution, in which the electrical density at each point is equal to the difference of the electrical densities in those two, would produce no force at external points. But, by the theorem alluded to, no distribution on a closed surface of any form can have the property of producing no force on external points; and therefore the hypothesis that there are two distinct solutions is impossible.

The theorem made use of in this reasoning is susceptible of special *analytical* demonstration (with the aid of the method in which "Laplace's coefficients" are employed) for the case of a spherical surface; but such an investigation would be inconsistent with the synthetical character of the present series of papers, and I therefore do no more at present than allude to the general theorem.

the mutual force between m at M and $-\frac{a}{f}m$ at I, a force which is necessarily attractive as the two electrical points M and I possess opposite kinds of electricity; we obtain

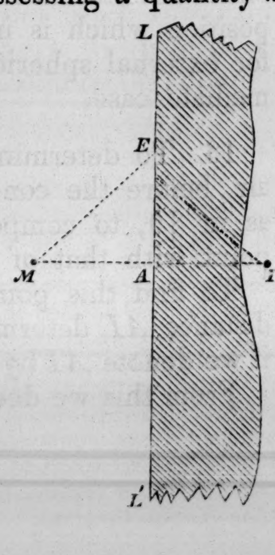
$$F = \frac{\frac{a}{f}m.m}{\left(\frac{a^2}{f} - f\right)^2} = \frac{afm^2}{(a^2 - f^2)^2} \ldots\ldots\ldots\ldots (B)$$

as the expression for the required attraction.

Electrical Influence on a Plane Conducting Surface of infinite extent.

11. If, in either the case of an external or the case of an internal spherical conducting surface, the radius of the sphere be taken infinitely great, the results will be applicable to the present case of an infinite plane; and it is clear that from either we may deduce the complete solution of the problem of determining the distribution of electricity, produced upon a conducting plane, by the influence of an electrical point. The "supplementary distribution," which in the case of a convex spherical conducting surface must in general be taken into account, will, in the case of a sphere of infinite radius, be a finite quantity of electricity uniformly distributed over a surface of infinite extent, and will therefore produce no effect; and the same results will, as is readily seen, be obtained whether we deduce them from the case of an external or of an internal spherical surface.

12. Let M be an electrical point possessing a quantity m of electricity placed in the neighbourhood of a conductor bounded on the side next M by a plane KK' which we must conceive to be indefinitely extended in every direction; it is required to determine the electrical density at any point E of the conducting surface.



Draw MA perpendicular to the plane, and let its length be denoted by p. We may, in the first place, conceive that instead of the plane surface we have a spherical conducting surface entirely enclosing the air in which M is insulated; and, supposing the shortest line from M to the spherical surface to be

equal to p, we should have, according to the notation of § 8,

$$f = a - p.$$

Hence the expression (A) becomes

$$\rho = -\frac{2ap - p^2}{4\pi a} \cdot \frac{1}{ME^3} = -\left(\frac{p}{2\pi} - \frac{p^2}{4\pi a}\right)\frac{1}{ME^3}.$$

In this, let a be supposed to be infinitely great; the second term within the vinculum will vanish, and we shall have simply

$$\rho = -\frac{p}{2\pi} \cdot \frac{1}{ME^3} \ldots\ldots\ldots\ldots\ldots (A)$$

for the required electrical density at the point E of the infinite plane electrified inductively through the influence of the point M.

Cor. The total amount of the electricity produced by induction is equal in quantity, but opposite in kind, to that of the influencing point M. We have seen already that the same proposition is true in general for internal spherical surfaces inductively electrified; but it does not hold for an external spherical surface, even if we neglect the "supplementary distribution, as it appears from the demonstration of IV. § 4, that the amount of the distribution expressed by the first term (that which varies inversely as the cube of the distance from the influencing point) of the value of ρ in equation (A) of IV. § 5, is equal to $-\frac{a}{f} m$. The infinite plane may, as we have seen, be regarded as an extreme case of either an external or an internal spherical surface; and the proposition which is in general true for internal, but not true for external spherical surfaces, holds in this limiting intermediate case.

13. To determine the resultant force at any point in the air, before the conducting plane, it will be only necessary, as in § 9, to compound the action of the given electrical point with that of an imaginary point I.

To find this point, we must produce MA beyond A to a distance AI, determined by the equation $CM.CI = a^2$; which, if we denote AI by p', becomes $(a - p)(a + p') = a^2$.

From this we deduce

$$p' = \frac{ap}{a - p} = \frac{p}{1 - \frac{p}{a}};$$

and thence, in the case of $a = \infty$, we deduce

$$p' = p.$$

Again, for the quantity of electricity to be concentrated at I, we have the expression

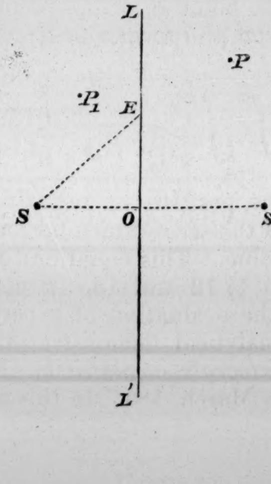
$$m' = -\frac{a}{a-p}m, \text{ or, when } a = \infty,\ m' = -m.$$

Hence the force at any point before the plane will be obtained by compounding that due to the given electrical point M, with a force due to an imaginary point I, possessing an equal quantity of the other hand of electricity, and placed at an equal distance behind the plane in the perpendicular MA produced.

14. If reference be made to the general demonstration (IV. § 2) on which all the special conclusions with reference to the effects of electrical influence on convex, concave, or plane conducting surfaces depend, we see that the geometrical construction employed fails in the case of a sphere of infinite radius, becoming nugatory in almost every step: we have however deduced *conclusions* which are not nugatory, but, on the contrary, assume a remarkably simple form for this case; and we may regard as rigorously established the solution of the problem of electrical influence on an infinite plane which has been thus obtained.

15. It is interesting to examine the nugatory forms which occur in attempting to apply the demonstrations of IV. §§ 2 and 4, to the case of an infinite plane; and it is not difficult to derive a special demonstration, free from all nugatory steps, of the following proposition.

Let LL' be an infinite "material plane," of which the "density" in different positions varies inversely as the cube of the distance from a point S, or from an equidistant point S_1, on the other side of the plane. The resultant force at any point P is the same as if the whole matter of the plane were concentrated at S; and the resultant force at any point P_1, on the other side of the plane, is the same as if the whole matter were collected at S_1.



16. In the course of the demonstration (in that part which corresponds to

the investigation in IV. § 4) it would appear that, if the density at any point E of the plane is given by the expression

$$\rho = \frac{\lambda}{SE^3},$$

the entire quantity of matter distributed over the infinite extent of the plane is given by the expression

$$m = \frac{2\pi\lambda}{p}.$$

This proposition and that which precedes it* contain the simplest expression of the mathematical truths on which the

---

* The two propositions may be analytically expressed as follows:—

Let O, the point in which SS_1 cuts the plane, be origin of coordinates, and let this line be axis of z. Then, taking OX, OY in the plane, let the coordinates of P be (x, y, z). Let also those of E be $(\xi, \eta, 0)$; so that we have

$$\rho = \frac{\lambda}{(\xi^2 + \eta^2 + p^2)^{\frac{3}{2}}}.$$

Hence the proposition stated in the text (§ 16), that the entire quantity of matter distributed over the infinite extent of the plane is equal to $\frac{2\pi\lambda}{p}$, is thus expressed:—

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\lambda\, d\xi\, d\eta}{(\xi^2 + \eta^2 + p^2)^{\frac{3}{2}}} = \frac{2\pi\lambda}{p}.$$

This equation may be very easily verified, and thus an extremely simple analytical demonstration of one of the theorems enunciated above is obtained.

Again, the proposition with reference to the attraction of the plane may, according to the well-known method, be expressed most simply by means of the potential. This must, in virtue of the enunciation in § 15, be equal to the potential due to the same quantity of matter, collected at the point S, or the point S_1, according as the attracted point is separated from the former or from the latter by the plane. Hence we must have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\lambda\, d\xi\, d\eta}{(\xi^2 + \eta^2 + p^2)^{\frac{3}{2}} \{(x - \xi)^2 + (y - \eta)^2 + z^2\}^{\frac{1}{2}}} = \frac{\frac{2\pi\lambda}{p}}{\{x^2 + y^2 + (\pm z + p)^2\}^{\frac{1}{2}}},$$

the positive or negative sign being attached to z in the denominator of the second member, according as z is given with a positive or negative value. This equation (of which a geometrical demonstration is included in §§ 12 and 13, in connection with IV. § 2) is included in a result, (the evaluation of a certain multiple integral) of which three different analytical demonstrations were given in a paper *On certain Definite Integrals suggested by Problems in the Theory of Electricity*, published in March, 1847, in this *Journal*, vol. II. p. 109.

solution of the problem of electrical influence on an infinite plane depends, and we might at once obtain from them the results given above. For an isolated investigation of this case of electrical equilibrium, this would be a better form of solution: but I have preferred the method given above, as the solution of the more general problem, of which it is a particular case, had been previously given.

17. The case of electrical influence which has been considered might at first sight appear to be of a singularly unpractical nature, since in strictness a conductor presenting on one side a plane surface of infinite extent in every direction would be required for realising the prescribed circumstances. If however we have a plane table of conducting matter, or covered with a sheet of tinfoil, or if we have a wall presenting an uninterrupted plane surface of some extent, the imagined circumstances are, as we readily see, *approximately* realized with reference to the influence of any electrical point in the neighbourhood of such a conducting plane, provided the distance of the influencing point from the plane be small compared with its distance from the nearest part where the continuity of the plane surface is in any way broken.

Fortbreda, Belfast, Oct. 17th, 1849.

MATHEMATICAL NOTES.

I.—*On the different published Demonstrations of "Pascal's Hexagramme."*

By Thomas Weddle.

The references for proofs (*by coordinate methods*) of Pascal's Hexagramme, communicated by me to the Editor of this Journal, and by him inserted as a foot-note at p. 285 of the last volume, were hastily written, partly from memory, without consulting any books but what I happened to have by me, and without any view to publication; it is not very surprising, therefore, that some omissions and inaccuracies should occur: the chief of these are the following.

A proof by Sir John W. Lubbock in the *Philosophical Magazine* for October, 1829, and another by Mr. Davies in the same periodical for July, 1842, are omitted.

The two solutions by Mr. Fenwick, in the *Diary* and *Davies's Hutton*, are the same; and it was *not* communicated to the *Philosophical Magazine*.

The account of the demonstrations in the *Lady's and Gentleman's Diary* is defective; but, previous to giving a complete one, I must premise that Pascal's theorem was proposed as the Prize Question in the *Diary* for 1842, to be proved 'without the aid of perspective, or any use of the particular case of the circle'; and in the number for next year there appeared six solutions, the first and third of which are geometrical, the others are by coordinate methods. I extract the headings of the different demonstrations.

The first solution is by "J. B. B. C., of Bristol."

"Second solution by Messrs. Thompson and Weddle, of Newcastle-upon-Tyne; and in nearly the same manner by Mr. Timothy Turbill, of Manchester."

"Third solution by Pen-and-Ink."

"Fourth solution by Professor Gill, College Point, New York."

"Fifth solution by Mr. Stephen Fenwick, of the Royal Military Academy, Woolwich, and similarly by Mr. G. W. Hearn, London."

"Sixth solution by Dunelmensis of Durham; and *M.* Chris. Stähelin, of Basle, Switzerland; principally from a solution by Magnus, a celebrated German mathematician."

I may here also remark, that the Rev. T. P. Kirkman has recently contributed a solution (*Lady's and Gentleman's Diary* for 1849, p. 85).

These demonstrations, added to those mentioned at p. 285, vol. III., include all the *coordinate* investigations with which I am acquainted. I believe the account will now be found pretty complete as far as English works are concerned; but it must, in justice both to the reader and myself, be understood that my search after investigations has not been very extensive, so that possibly some may still be overlooked.

Wimbledon, April 9, 1849.

*II.—Comparison of Expressions for Circular and Elliptic Functions in Continued Fractions.**

By C. J. MALMSTÉN, Professor of Mathematics in the University of Upsala.

THE following formula is already known.
See Schlömich's *Analys.*, p. 341; Lacroix, *Traité d. Cal. Diff. et Integr.*, tom. II. p. 432.

$$\frac{x}{2}\tan^{-1}(2x) = \cfrac{x^2}{1 + \cfrac{2.2.x^2}{3 + \cfrac{4.4.x^2}{5 + \cfrac{6.6.x^2}{7 + \cfrac{8.8.x^2}{9 + \cfrac{10.10.x^2}{11 + \text{etc. } in\ inf.}}}}}}$$

On account of the remarkable resemblance in structure, the following deserves to receive a place beside it.

$$(1 - 2u^2)\left\{\frac{E'(u)}{F'(u)} - \frac{1 - u^3}{1 - 2u^2}\right\} = \cfrac{x^2}{2 + \cfrac{3.3.x^2}{4 + \cfrac{5.5.x^2}{6 + \cfrac{7.7.x^2}{8 + \cfrac{9.9.x^2}{10 + \text{etc. } in\ infin.}}}}}$$

where $u = \sqrt{\left\{\frac{1}{2}\left(1 - \frac{1}{\sqrt{(1 + x^2)}}\right)\right\}}$,

and E', F' compl. ellipt. functions of the second and first kind, respectively.

[The demonstration of this theorem has been communicated by Prof. Malmstén for publication in Crelle's Journal.]

III.—Sur l' Intégration des Equations Différentielles Linéaires.

Par C. J. MALMSTÉN.

ÉTANT données $n - 1$ solutions particulières d'une équation différentielle linéaire du n^ième^ ordre, il reste à trouver encore une solution particulière pour en former l'intégrale générale. On sait bien, que cette solution restante se trouvera à l'aide d'une équation linéaire du 1^er^ ordre; mais déjà pour les équations du 3^ième^ et 4^ième^ ordre les calculs semblent devenir

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* [This "Mathematical Note," and the two which follow it, have been communicated by the Author for publication in the *Mathematical Journal*. They are abstracts of papers contributed to *Crelle's Journal*, but not yet published.—*Upsala*, *Sept.* 17, 1849.]

si compliqués, qu' on a même cru qu' il ne vaudrait pas la peine les conduire au bout; à fin d' obtenir l' expression finale de la valeur démandée. Cependant nous avons réussi à éluder cette difficulté à l'aide des fonctions connues sous le nom de *Déterminantes;* qui nous ont mis en état de présenter l' expression cherchée sous une forme très simple.

Le Théorème général, auquel nous sommes parvenus, est le suivant:

Soient $$y_1, y_2, y_3, \dots y_{n-1},$$

$n-1$ *intégrales particulières de l' équation*

$$\frac{d^n y}{dx^n} + P \cdot \frac{d^{n-1}y}{dx^{n-1}} + Q \cdot \frac{d^{n-2}y}{dx^{n-2}} + \dots + S \cdot \frac{dy}{dx} + Ty = 0,$$

cette équation sera aussi satisfaite par

$$y_n = y_1 . z_1 + y_2 . z_2 + y_3 . z_3 + \dots + y_{n-1} . z_{n-1},$$

où l' on a en général

$$z_r = (-1)^{n-1} \cdot \int \frac{dR}{dy_r^{(n-2)}} \cdot e^{-\int P dx} . dx.$$

et $$\frac{1}{R} = \Sigma \pm y_1 . y_2' . y_3'' \dots y_{n-1}^{(n-2)},$$

en désignant par $y_r^{(n)}$ *la* $n^{ième}$ *dérivée de* y_r.

Ex. 1. Pour $n = 2$, c' est à dire pour l' équation

$$\frac{d^2y}{dx^2} + P \cdot \frac{dy}{dx} + Q . y = 0.$$

Nous supposons y_1 connu; donc nous aurons

$$z_1 = \int \frac{e^{-\int P dx}}{y_1^2}\, dx, \text{ et partant } y_2 = y_1 \cdot \int \frac{e^{-\int P dx}}{y_1^2}\, dx,$$

d' où l' intégrale complète sera

$$y = K_1 . y_1 + K_2 . y_1 \cdot \int \frac{e^{-\int P dx}}{y_1^2}\, dx,$$

étant K_1 et K_2 deux constantes arbitraires.

Ex. 2. Pour $n = 3$, c. à d. pour l' équation

$$\frac{d^3y}{dx^3} + P \cdot \frac{d^2y}{dx^2} + Q \cdot \frac{dy}{dx} + R . y = 0.$$

Nous supposons y_1 et y_2 (deux valeurs particulières) connues; donc nous aurons

$$z_1 = \int \frac{y_2 e^{-\int P dx} . dx}{(y_1 . y_2' - y_2 . y_1')^2}, \quad z_2 = -\int \frac{y_1 e^{-\int P dx} . dx}{(y_1 . y_2' - y_2 . y_1')^2},$$

et partant

$$y_3 = y_1 \cdot \int \frac{y_2 e^{-\int P dx} . dx}{(y_1 . y_2' - y_2 . y_1')^2} - y_2 \cdot \int \frac{y_1 e^{-\int P dx} . dx}{(y_1 . y_2' - y_2 . y_1')^2},$$

d'où l'intégrale complète

$$y = K_1 y_1 + K_2 y_2 + K_3 \left\{ y_1 \cdot \int \frac{y_2 e^{-\int P dx} . dx}{(y_1 . y_2' - y_2 . y_1')^2} - y_2 \cdot \int \frac{y_1 e^{-\int P dx} . dx}{(y_1 . y_2' - y_2 . y_1')^2} \right\}.$$

Ex. 3. Pour $n = 4$, c. à. d. pour l'équation

$$\frac{d^4 y}{dx^4} + P . \frac{d^3 y}{dx^3} + Q . \frac{d^2 y}{dx^2} + R . \frac{dy}{dx} + S . y = 0.$$

Nous supposons les trois valeurs particulières y_1, y_2, et y_3 connues; donc nous aurons

$$z_1 = \int \frac{(y_2 . y_3' - y_3 . y_2') e^{-\int P dx}}{N^2} dx, \quad z_2 = - \int \frac{(y_1 . y_3' - y_3 . y_1') e^{-\int P dx}}{N^2} dx,$$

$$z_3 = \int \frac{(y_1 . y_2' - y_2 . y_1') e^{-\int P dx}}{N^2} dx,$$

et partant

$$y_4 = y_1 \cdot \int \frac{(y_2 . y_3' - y_3 . y_2') e^{-\int P dx}}{N^2} dx - y_2 \cdot \int \frac{(y_1 . y_3' - y_3 . y_1') e^{-\int P dx}}{N^2} dx$$

$$+ y_3 \cdot \int \frac{(y_1 . y_2' - y_2 . y_1') e^{-\int P dx}}{N^2},$$

d'où l'intégrale complète

$$y = K_1 y_1 + K_2 y_2 + K_3 y_3 + K_4 y_4,$$

en posant, pour abréger les expressions,

$$N = y_1 (y_2' . y_3'' - y_3' . y_2'') + y_2 (y_3' . y_1'' - y_1' . y_3'') + y_3 (y_1' . y_2'' - y_2' . y_1'').$$

END OF VOLUME IV.

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